KU LEUVEN

FACULTY OF SCIENCE

# The holonomy groupoid of foliations

## Master thesis

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# Foreword

The aim of this thesis is to introduce and study the notion of holonomy of foliations. To this end, we devote a large portion of our time studying the basic theory surrounding regular and singular foliations, groupoids, algebroids and several other related topics. Our first objective is to construct the holonomy groupoid of regular foliations, which is a classical object. Afterwards, we give an introduction to the holonomy groupoid as given by Androulidakis and Skandalis in their pioneering paper [AS], and we use the notion of holonomy transformations, introduced by Androulidakis and Zambon in [AZ], to give this groupoid a geometric point of view. Through the use of lots of examples and figures, we try to deepen the understanding of the reader.

I would like to devote this section of the thesis to thank the wonderful people, without whom this thesis wouldn't be possible.

- First of all, I would like to acknowledge my supervisor professor Marco Zambon. His patience and his role as a guide were crucial not only for the development of this thesis, but for my growth as a mathematics student. My admiration and thankfulness towards him cannot be overstated.
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- I would like to thank my high school teachers, Guy Van Reeth and Veerle Pues for inspiring me, and also introducing me to the beautiful world of mathematics.

# Symbols

| | |
|---|---|
| $C^\infty(M, E)$ or $\Gamma(E)$ | Sections of vector bundle $E$ |
| $\mathcal{F}$ | A foliation |
| $\mathfrak{g}$ | A Lie algebra |
| $F_x$ | Leaf through x |
| $X \times_{f,g} Y$ | Fibered product |
| $\#$ | The anchor map of a Lie algebroid |
| $\mathfrak{X}(M)$ | The vector fields on $M$ |
| $G \rightrightarrows M$ | Lie groupoid |
| $M/\mathcal{F}$ | The leaf space |
| $x \sim y$ | $x$ is related to $y$ with respect to some equivalence relation |
| $[x]$ | the equivalence class of $x$ with respect to some equivalence relation |
| $G_x$ | The isotropy group of $x$ |
| $G \curvearrowright M$ | An action of $G$ on $M$ |
| $M \times N$ | The product manifold from $M$ and $N$ |
| $\alpha : x \mapsto y$ | A path $\alpha$ from $x$ to $y$ |
| $\|X\|$ | The cardinality of the set $X$ |
| $1_x$ | The unit arrow at $x$ |

# Contents

# Introduction

We divide the introduction into three parts: The motivation for the topics in this thesis, stating the problems we will discuss and finally giving a sketch of how we will approach these problems.

## Motivation

Central to this thesis is the notion of foliations. Foliations arose from the study of differential equations, where they were used to qualitatively study problems in the field. Since then, foliations have become a field on their own (often referred to as foliation theory) and are intensively studied in modern day mathematics. Their applications are widespread, for example in Poisson geometry, differential equations, optimal control theory, geometric mechanics and Lie group actions. We distinguish two types of foliations, regular foliations and singular foliations. The first two chapters are devoted to discussing regular foliations, whilst the remainder looks at the singular case. Intuitively, a regular foliation is a decomposition of a smooth manifold $M$ into immersed submanifolds (called the leaves), which fit together nicely in the manifold. The concrete definition is given in definition 1.1. The decomposition of our manifold can be locally modelled by the affine decomposition of $\mathbb{R}^n$ into copies of $\mathbb{R}^k$, see figure 1.

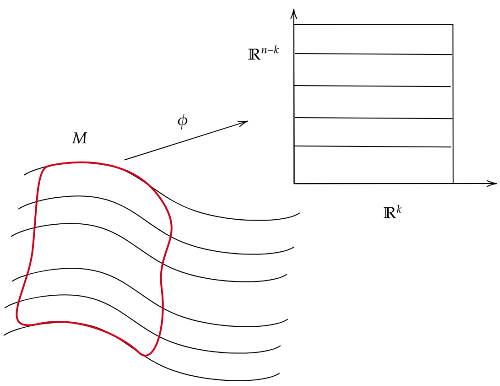



Figure 1: The local picture of a foliation

Despite the fact that they have such a nice local interpretation, their global structure can be quite involved. To be able to understand foliations a bit better, the notion of the holonomy groupoid comes into play. The holonomy groupoid is the focal point of this thesis. In the case of regular foliations, it is a classical object dating back to the works of Ehresmann in 1965 ([Ehr]). To motivate why this groupoid is important, we highlight two applications. For the first one, as we will see in this thesis, the holonomy groupoid encodes a lot of geometric information regarding the foliation. For example, it tells us how the leaves of a foliation wrap around each other globally. Another example comes from the fact that a foliation can be seen as an equivalence relation on the manifold with some smoothness criteria. In general, this equivalence relation (seen as the set of pairs $(x, y)$ with $x \sim y$) is not a smooth submanifold of the product manifold $M \times M$. The holonomy groupoid can be used as a model of the equivalence relation, and is always smooth. These give a motivation from a purely geometric point of view. The second application (as mentioned in [Gd], pp.3) lies in the field of non-commutative geometry. Recall that by a theorem of Gelfand, one can model a commutative $C^*$-algebra on the space of functions of a topological space $X$. In the non-commutative setting, this no longer need to hold true. The upshot of the holonomy groupoid is that one can associate to them a $C^*-$algebra that in general fails to be commutative, which yields a more geometric interpretation of these $C^*-$algebras.

# Problem statement

Motivated by the applications of the holonomy groupoid, we will tackle the following three problems.

1. How is the holonomy groupoid of a regular foliation constructed?

2. Can this construction be generalised to the more general case of singular foliations?

3. Is there a way to geometrically interpret these holonomy groupoids?

# Approach

Before tackling any of these problems, we will give a soft introduction to foliation theory. Not only does this give the reader more context, it also supplies us with tools and examples that are crucial to understand the problems at hand. This is the content of chapter 1. Chapter 2 is devoted to constructing the holonomy groupoid for regular foliations, by giving the necessary background information. In chapter 3, we introduce the reader to singular foliations. By giving some basic results, we familiarise the reader with some important properties that they possess. In chapter 4, we will discuss the construction of the holonomy groupoid as given by Androulidakis and Skandalis in [AS]. We guide the reader through the construction, motivating the steps taken along the way. Finally, in chapter 5, we tackle the final problem. Using the notion of holonomy transformations, as introduced by Androulidakis and Zambon ([AZ]), we show a way to geometrically think about the holonomy groupoid.

# Chapter 1

# Regular Foliations

In this chapter, we discuss the basics of foliation theory in the regular case. We devote a large portion of the chapter on giving examples and constructions. Afterwards, we will talk about the holonomy of foliations, which measure in some sense how much the leaves twist. We will also briefly discuss the topology of the leaves. Good references are [MM], [Lee] and [CC]. For the section on the topology of the leaves, a good reference is [CN]. The introductory section is also based on written notes from the master course in Differential Geometry at KU Leuven during the first semester in 2018, given by professor Marco Zambon.

## 1.1 Definitions and basic results

### 1.1.1 Definitions

As mentioned in the introduction, a foliation is a nice partition of our manifold into immersed submanifolds.

**Definition 1.1.** Let $M$ be a smooth manifold of dimension $n$. Let $\{L_\alpha\}_{\alpha\in I}$ be a family of disjoint $k-$dimensional, connected and immersed submanifolds of $M$. We say that $\{L_\alpha\}_\alpha$ is a rank $k$ (regular) foliation of $M$ if the following conditions are satisfied.

1. $\coprod_\alpha L_\alpha = M$, i.e $M$ is covered by the $L_\alpha$.
2. For each point $p \in M$, there is a chart $(U, \phi = (x_1, ..., x_n))$ at $p$ such that for each $L_\alpha$, the set $U \cap L_\alpha$ is either empty, or

$$\phi(L_\alpha \cap U) = \bigcup_{i\in I} \left\{x_{k+1} = \alpha^i_{k+1}, ..., x_n = \alpha^i_n\right\},$$

   where $I$ is some countable index set and each $\alpha^i_j \in \mathbb{R}$ is a constant.

We denote a foliation by $\mathcal{F}$, and call the tuple $(M, \mathcal{F})$ a foliated manifold. We call the elements of $\{L_\alpha\}$ the leaves of the foliation.

In other words, a foliation is a partition of our manifold such that locally, this partition looks like the affine partition of $\mathbb{R}^n$ into $\mathbb{R}^k-$dimensional subspaces. In this point of view, one should interpret a foliation as copies of $\mathbb{R}^k$ 'stacked up together' in some bigger ambient Euclidean space $\mathbb{R}^n$. Despite being very intuitive, this definition is often not that handy when doing computations. An alternative definition is obtained via Frobenius' lemma (see Appendix 6.1).

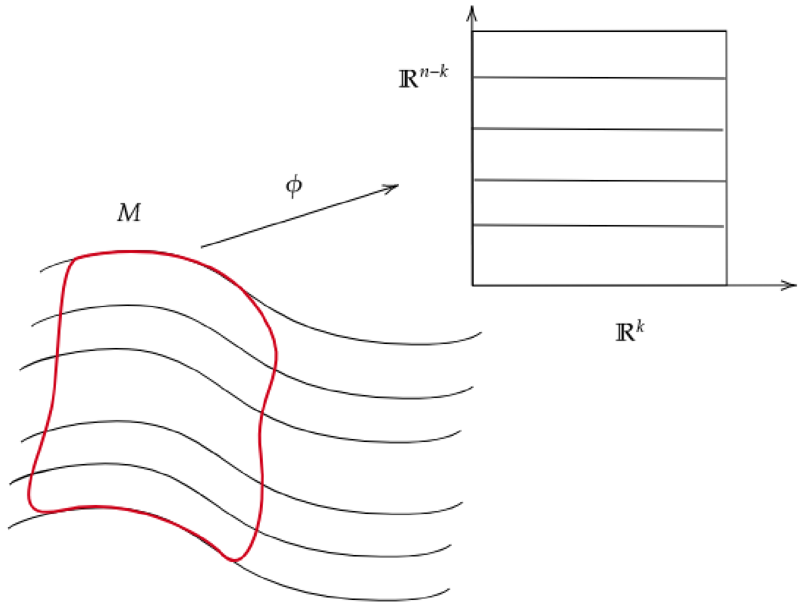



Figure 1.1: The local picture of a foliation

**Definition 1.2.** [Lee] A foliation $\mathcal{F}$ on a smooth manifold $M$ is an involutive distribution $D \subset TM$.

The final definition of regular foliations is of another flavour. It uses a special kind of atlas, called the foliation atlas that witnesses the identification of the foliation with the affine decomposition of $\mathbb{R}^n$. However, since a foliation is a global object, one needs to pay attention when considering overlaps of foliation charts.

**Definition 1.3.** [MM] Let $M$ be a smooth $n-$manifold. Suppose that we have an atlas

$$\left\{\phi_i : U_i \to \mathbb{R}^{n-k} \times \mathbb{R}^k\right\}_{i\in I}$$

such that the change of chart diffeomorphisms are of the form

$$\phi_{ij}(x,y) = \phi_i \circ (\phi_j|_{U_i \cap U_j})^{-1}(x,y) = (h_1(x,y), h_2(y))$$

with respect to the affine decomposition $\mathbb{R}^n = \mathbb{R}^{n-k} \times \mathbb{R}^k$. Remark that $h_2$ does not depend on $x$. Then we call this atlas the foliation atlas of $M$.

**Remark.** This gives us a decomposition of each $U_i$ into the connected components of the submanifolds $\phi_i^{-1}(\mathbb{R}^{n-k} \times \{y\}), y \in \mathbb{R}^k$. We call such connected components the plaques of $M$.

Following proposition ensures us that the condition on the overlap is sufficient.

**Proposition 1.1.** The change of chart diffeomorphisms respect the decomposition of each $U_i$ into plaques. in other words: the decomposition of two chart domains $U_i$ and $U_j$ into plaques coincide on their intersection.

*Proof.* Consider two local charts $(U, \phi)$ and $(V, \psi)$ in the foliation atlas. If $U \cap V = \emptyset$, there is nothing to prove. Hence, we may suppose $p \in U \cap V$. This will lie on two plaques, $L_U = \phi^{-1}(\mathbb{R}^{n-k}, y)$ and $L_V = \psi^{-1}(\mathbb{R}^{n-k}, y')$, where $y, y'$ are fixed. Write $p = \phi^{-1}(x, y) = \psi^{-1}(x', y')$. We have to show that $L_U \cap V = L_V \cap U$. Since $L_U, V$ are both open, we can take a point $q \in U \cap V$ such that $q \in L_U$. Recall that the transition function $\vartheta = \psi \circ \phi^{-1}$ is of the form $(h_1(x, y), h_2(y))$. Hence, $\pi_2 \circ \vartheta(p) = \pi_2 \circ \vartheta(q) = h_2(y)$. Thus, $y' = z' = h_2(y)$, meaning that $q \in L_V$. Analogously, one can show that $L_V \cap U \subset L_U \cap V$, which shows the proposition. □

Hence, we can glue together the (locally defined) plaques into a well-defined global decomposition of $M$ in the following way.

**Definition 1.4.** Let $M$ be a smooth manifold, endowed with a foliation atlas $(U_i, \phi_i)$. Equip it with an equivalence relation on $M$ defined as follows: $p \sim q$ if and only if there is a sequence of plaques $\alpha_1, ..., \alpha_k$ with $p \in \alpha_k$ and $q \in \alpha_k$, where $\alpha_i \cap \alpha_{i+1} \neq \emptyset$ for all $i$. We call the equivalence classes of this relation the leaves of the foliation chart.

Suppose we wish to capture this equivalence relation using the natural set

$$\mathcal{R} = \{(x, y) | x \sim y\} \subset M \times M.$$

Evidently, this captures all the information of the foliation. However, in general, it fails to be smooth (for an example of this fact, we refer to [Ph]). This huge drawback will lead us to the notion of holonomy which we will define later. This generalises above idea, but in the case of regular foliations is always smooth. Another interesting object is the leaf space of a foliated manifold $(M, \mathcal{F})$. This is the quotient $M/\mathcal{F}$, which we endow with the quotient topology. This topological space can have very messy topology and sometimes recovers barely any information.

**Remark.** Being an immersed submanifold implies that each leaf is a fortiori also a topological space. The topology of the leaves is the one generated by taking plaques as basis opens. We will see later that this topology need not coincide with the subspace topology of the ambient manifold. We now consider the smooth structure of the leaves. For this, we use ([CN],pp. 31). Given a point $p$ on a leaf $F$, choose a foliation chart $(U, \phi)$ around $p$. Denote by $L_p$ the plaque of the foliation chart containing $p$. Like before, we can then write $L_p = \phi^{-1}(\mathbb{R}^k \times y)$ for some fixed $y$. Denote by $\phi_1$ the map $\pi_1 \circ \phi$. Here $\pi_1$ is the projection map $\pi_1 : \mathbb{R}^k \times \mathbb{R}^{n-k} \mapsto \mathbb{R}^k$. It is easy to see that $\phi_1|_{L_p} : L_p \to \mathbb{R}^k$ is a homeomorphism onto its image. The set

$$\mathcal{U} = \{(L, \phi_1) | (U, \phi) \text{ is a foliation chart}, L \subset F \text{ is a plaque}\}$$

defines a smooth structure for the leaf $F$. For more details, we refer to the source material ([CN], pp.31-32).

### 1.1.2 Examples of Regular Foliations

Having an arsenal of examples is often almost as important as knowing the definition itself. Therefore, we devote a section on listing some important examples (and non-examples) of regular foliations.

**Example 1.1.** In some sense the easiest of foliations is given by the affine decompositions of Euclidean spaces. An evident family of foliations on $\mathbb{R}^\kappa$ is given by

$$\mathbb{R}^n = \cup_{x \in \mathbb{R}^{n-k}} \mathbb{R}^k \times \{x\}$$

where for example the identity map generates a foliation atlas. Since any vector space automorphism $\theta$ of $\mathbb{R}^n$ respects affine decompositions, this brings forth a new class of (trivial) examples.

**Example 1.2.** Another trivial example is the foliation of a smooth manifold $M$ by points. The associated distribution is the zero distribution. Notice that, in particular, every manifold can be foliated (although not necessarily by a particularly interesting foliation). On the other extreme, any manifold $M$ can also be foliated by the one-leaf foliation. This is the foliation whose leaf is $M$.

**Example 1.3.** [MM] Let $f : M^m \to N^n$ be a submersion. This induces a foliation $\mathcal{F}(f)$ on $M$ as follows. By the submersion theorem, there are local coordinates such that $f$ is a projection of $\mathbb{R}^m$ onto $\mathbb{R}^n$. Fibers of such projection maps give rise to an affine decomposition as in example 1.1. Hence, the fibers of a submersion locally look like the desired decomposition. One can show that the partition whose leaves are the fibers of $f$ is indeed a foliation.

**Example 1.4.** [CC] Consider the partition of the square into lines of slope $a$. Since this partition is invariant under translations, this induces a partition of the torus $T^2$ by lines as seen in figure 1.2. One can show that this partition is a foliation on $T^2$, called the Kronecker foliation. The geometry of the leaves is highly dependent on the rationality of $a$. If $a$ is rational, one can evidently see that the leaves form closed loops on the torus, hence they are compact. The topology on the leaf is equivalent with the subspace topology so that they are in fact embedded submanifolds. Being a compact, connected one-dimensional manifold, they are diffeomorphic to $S^1$. In the case where $a$ is irrational, the leaf winds infinitely many often around the torus, never closing up. In this case, they are dense in $T^2$ and are diffeomorphic to $\mathbb{R}$. In particular, they are not compact. Notice that in this case, the leaves are not embedded submanifolds! Indeed, a manifold has to be locally path-connected which is not the case for the leaves in the topology of the torus.

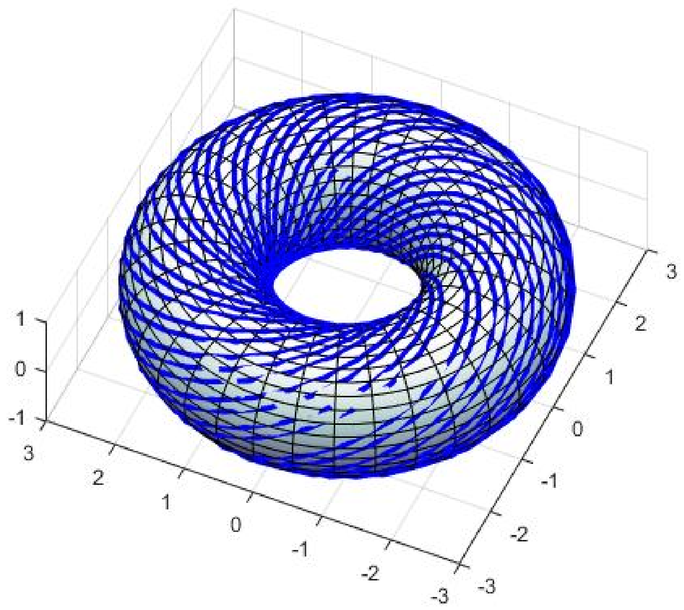



Figure 1.2: The Kronecker foliation on the torus

**Example 1.5.** [Lee] Consider the distribution $D = \{\partial_y, \partial_x + y\partial_z\}$, see figure 1.3. This does not define a foliation of $\mathbb{R}^3$, since it is not an involutive distribution. This distribution is, in fancy terms, an example of a contact distribution on $\mathbb{R}^3$. For a geometric interpretation of the involutivity condition, we recall following result.

**Lemma 1.1.** [Lee] Let $M$ be a smooth manifold, $N \subset M$ an immersed submanifold. Let $V$ be a smooth vector field tangent to $N$. Then any integral curve $\gamma$ of $V$ such that $\gamma(0) \in N$, we can find an $\epsilon > 0$ such that $\gamma((-\epsilon, \epsilon)) \subset N$.

Hence, suppose $L$ is a leaf of the distribution $D$ through the origin. Then, flowing along $\partial_x + x\partial_z$ starting from the origin traces out a piece of the $x-$axis. By above result, there is a small piece in the $x-$axis that must be contained in $L$. Flowing along $\partial_y$, starting from these points on the $x-$axis, traces out a small piece of the $(x, y)-$plane. However, this implies that $L$ cannot be an integral manifold.

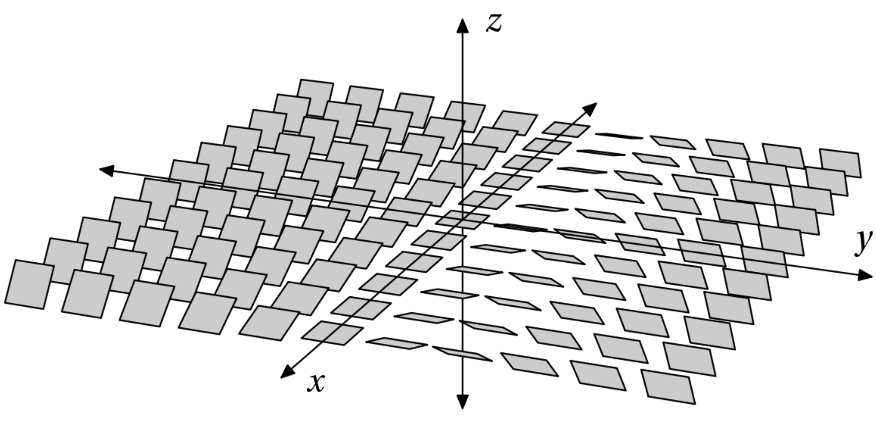



Figure 1.3: The standard contact structure on $\mathbb{R}^3$

**Example 1.6.** [MM] A rich source of examples comes from Lie group actions. It is a general result that the orbits $Gx$ of Lie group actions can be seen as immersed submanifolds of $M$. One problem is that in general these orbits are not equidimensional. An easy way to go around this problem is by restricting our attention to so-called *foliated* Lie group actions, which are actions such that $\dim(G_x)$ is constant for $x$.
As an example, consider the action of $S^1$ on the punctured plane $\mathbb{R}^2 \setminus \{(0,0)\}$. The orbits are concentric circles. To see that it is a foliation, notice that the infinitesimal generator of this action is $y\partial_x - x\partial_y$, which gives rise to an involutive distribution on $M$.

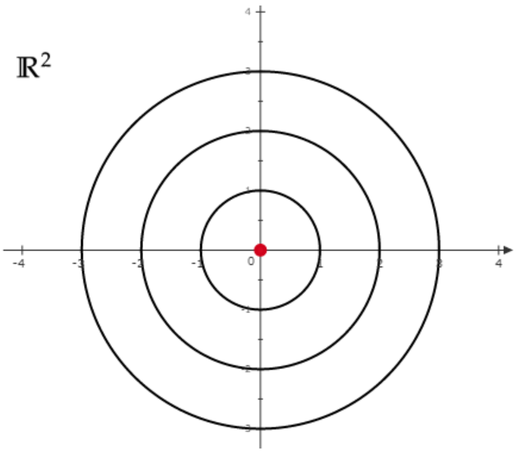



Figure 1.4: The orbits of the action of $S^1$

**Example 1.7.** Consider the Möbius band $M$, foliated by the action of $S^1$ obtained by wrapping $S^1$ around $M$ twice, except on the central circle. This is sketched in figure 1.5. This is easily seen to be a foliated Lie group action. This example will be used frequently later on.

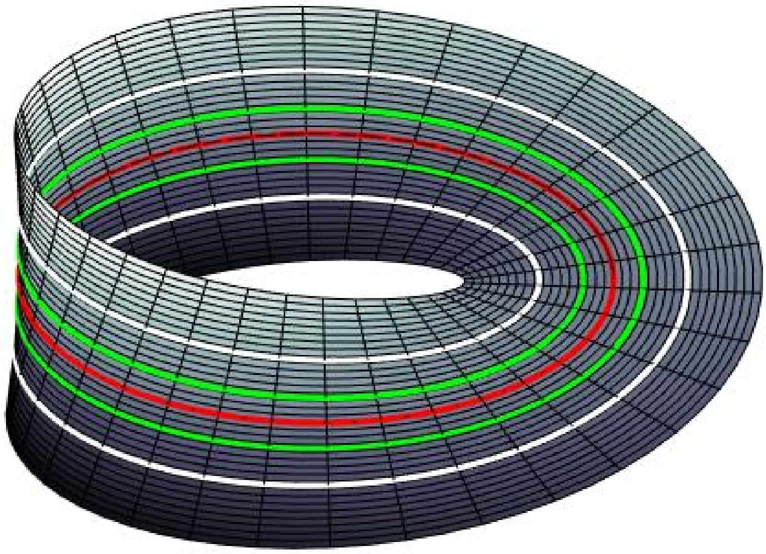


Figure 1.5: The foliation on the Mobiüs band

**Example 1.8.** [MM] Another rich source of examples comes from so-called quotient foliations. Suppose we have a manifold $M$ with a foliation $\mathcal{F}$. Suppose a group $G$ acts on $M$ freely, properly discontinuous and by diffeomorphisms. In this case, from the quotient manifold theorem ([Lee], thm. 21.10), we obtain a manifold $M/G$.
A sufficient condition for a foliation $\mathcal{F}$ on $M$ to drop down to $M/G$ is that the action leaves the leaves invariant, in the sense that leaves get mapped to leaves. In this case, we get a foliation of $M/G$ whose leaves are diffeomorphic to the quotient $L/G_L$, where $G_L$ is the isotropy of $L$, defined as

$$G_L = \{g \in G | g(L) \subset L\}.$$

A special type of quotient foliations is given by suspensions of diffeomorphisms. The idea is as follows: Suppose we have a diffeomorphism $f : M \to M$. Consider the natural vector bundle $\mathbb{R} \times M$ over $M$. This can be foliated by leaves of the form $\mathbb{R} \times \{x\}$ for each $x$. Define the following action of $\mathbb{Z}$ on $\mathbb{R} \times M$;

$$\mathbb{Z} \curvearrowright \mathbb{R} \times M : (z, (r, x)) \mapsto (r + z, f^z(x)).$$

Since $f$ is a diffeomorphism, $f^z$ makes sense for all integers $z$. Evidently, $f$ maps leaves to leaves, and acts properly discontinuous. Hence, we get a quotient foliation on the manifold $(\mathbb{R} \times M)/\mathbb{Z}$. We will denote this manifold by $\mathbb{R} \times_{\mathbb{Z}} M$. Notice that this space is a fiber bundle over $S^1$, and the corresponding leaves are $1-$dimensional.
Notice that the foliation on the Möbius band is an example of this type, where the diffeomorphism is $-Id$.

**Example 1.9.** [MM] Suppose we have a smooth map $f : N \to M$, with $\mathcal{F}$ a foliation on $M$. One question is whether a map can pull back the foliation on $M$ to a foliation $f^*(N)$. The following definition gives a sufficient condition on $f$.

**Definition 1.5.** Let $f : N \to M$ be smooth. Suppose $U$ is an immersed submanifold of $M$. We say that $f$ is transversal to $U$ if for all $x$ in $N$, one has

$$(df)_x(T_x(N)) + T_{f(x)}U = T_xM.$$

We say that $f$ is transversal to $\mathcal{F}$ if it is transversal to every leaf it meets, and denote it by $f \pitchfork \mathcal{F}$.

One can show ([MM], pp. 14) that if $f$ is transversal to $\mathcal{F}$, the partition of $N$ by the connected components of the sets $f^{-1}(F)$ (where $F$ is a leaf of $\mathcal{F}$) is a foliation on $N$.

**Example 1.10.** [MM] In this example, we define the notion of flat bundles. The idea is to define a fiber bundle $E \xrightarrow{\pi} M$, and foliate $E$ by 'horizontal' leaves, i.e leaves that under $\pi$ get mapped (as covering projections) to $M$. Let us start by an illuminating example. Consider the Möbius band $E$ (see example 1.7 and figure 1.5), which can easily be seen as a fiber bundle over $S^1$. Endowing $E$ with the usual foliation, notice that each leaf gets mapped to $S^1$ under $\pi$ as a covering map. Every leaf except the central leaf are equivalent (as covering spaces) to the two-cover of $S^1$. The central leaf is easily seen to equivalent to the covering of $S^1$ by itself.

For the general case, let $M$ be a smooth manifold. Suppose there exists a connected manifold $\tilde{M}$, upon which a group $G$ acts freely and discontinuously, such that $\tilde{M}/G = M$. Suppose we have another manifold $F$ that also has an action of $G$. Then we can form the quotient space $E = \tilde{M} \times_G F$. This is obtained by identifying the actions, in the sense that we identify $(mg, f)$ with $(m, gf)$. By doing this, we can look at $E$ as if it was the orbit space of $\tilde{M} \times F$. One can show that $E$ is a smooth manifold.

To get a fibre bundle, notice that the projection $\pi_1 : \tilde{M} \times F \to \tilde{M}$ induces a submersion $\pi : E \to M$, and we have the following commutative diagram, where $q$ and $q'$ are the quotient maps.

$$\begin{array}{ccc} \tilde{M} \times F & \xrightarrow{q} & E \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi} \\ \tilde{M} & \xrightarrow{q'} & M \end{array}$$

We can then view $\pi : E \to M$ as a fiber bundle over $M$ with fibre $F$. In our example, we have that $\tilde{M} = \mathbb{R}$, $M = S^1$, $F = (-1, 1)$, $G = \mathbb{Z}$ and $E$ is the Möbius band. The $G-$action on $\tilde{M}$ comes from identifying $\mathbb{Z}$ with the fundamental group, and the $G-$action on $F$ is $(z, x) = (-1)^z x$.

The foliation of $\tilde{M} \times F$ given by $\tilde{M} \times \{x\}, x \in F$ is evidently invariant under $G-$action. Hence, we have a foliation $\mathbb{F}$ on $E$. Notice that the leaf obtained from $\tilde{M} \times \{z\}$ is naturally diffeomorphic to $\tilde{M}/G_z$, where $G_z$ is the isotropy of $z$. Furthermore, the restriction $\pi : \tilde{M}/G_z \to M$ is a covering projection of $M$.

### 1.1.3 Topology of Leaves

We have already seen that the topology of leaves need not coincide with the topology from the ambient manifold. In this section, we are going to focus on the leaves of a foliation.

When looking at the local model of foliations, there is a clear notion of 'transversal' and 'longitudinal' or 'leafwise'. We have yet to introduce a notion that captures the transversal information.

Suppose $\mathcal{F}$ is a foliation on $M$, and let $F$ be some fixed leaf of $\mathcal{F}$. Then the plaques of $F$ are of the form $\phi^{-1}(U_1 \times \{x\})$, for some foliation chart $(\phi, U_1)$. Similarly, one can define a transverse section of $F$ by $\phi^{-1}(y \times U_2)$. Notice that $T_pF \oplus T_pS = T_pM$, motivating the terminology.

As mentioned before, leaves are used for leafwise information, and transversal sections are used for transversal information. A stringent question is whether transversal sections capture *only* transversal information, in the sense that they capture the same information near some leaf $F$ regardless of position on the leaf.

Following proposition, aptly called *transversal uniformity*, ensures us that this is the case. The following proposition and proof can be found in ([CN], pp. 49).

**Proposition 1.2.** [CN] Let $(M, \mathcal{F})$ be a foliated manifold, and $F$ a leaf of $\mathcal{F}$. For each pair of points $p, q \in F$, we can find transverse sections $S, T$ at $p$ and $q$ respectively, together with a diffeomorphism $f : S \to T$ with the property that for every leaf $F'$, one has

$$f(F' \cap S) = F' \cap T.$$

*Proof.* Let $p, q$ be two points on the same leaf. By definition, there exists a sequence $(\alpha_1, U_1), ..., (\alpha_n, U_n)$ of foliation charts such that $U_i \cap U_{i+1} \neq \emptyset$ for each $i$, and so that $p \in U_1$ and $q \in U_n$. Since the intersection of subsequent chart domains is non-empty, we can fix a sequence of points $x_i$ in the intersection of the plaques $U_i \cap U_{i+1}$. We set $x_0 = p$, and we require $x_n = q$. For each of these points, we can choose a transversal disk $D_i$. Shrinking $D_i$ if necessary, we can assume that each $D_i$ intersects the plaques of $\alpha_{i+1}$ just once. This allows us to define a map $\phi : D_i \to D_{i+1}$, by mapping each point on $D_i$ to the unique point on $D_{i+1}$ lying on the same plaque. In local coordinates, this map is nothing more than a translation, and hence evidently smooth. Doing this for every disk, we get a diffeomorphism $\phi : D_0 \to D_n$ which by construction satisfies the intersection property. □

This result tells us what the global topology of transversal sections is allowed to look like, only knowing local information. What this doesn't say is that we should therefore not care about the global transversal behaviour of foliations. We motivate this by following example.

**Example 1.11.** Let $M$ be the Möbius band, endowed with its familiar foliation. Consider a transverse section $T$ of $\mathcal{F}$ at a point $p$ on the central circle. Let us consider the allowed automorphisms of $T$ that satisfy the intersection property in above proposition. One evident choice is the identity. Notice that $T$ intersects leaves close to the central circle twice, giving us two plaques $P_1^F$ and $P_2^F$ for each such leaf $F$.

Assume that $T$ is a transversal section that is symmetric, in the sense that it intersects either both plaques of the same leaf, or doesn't intersect the leaf at all. Then the diffeomorphism of $T$ that interchanges $P_1^F$ and $P_2^F$ also satisfies the intersection property, but is of a completely different nature. This diffeomorphism, intuitively, flips $T$ around its centre. This implies that there is some twisting in the transverse geometry. To see this, one loops $T$ around the Möbius ring as in figure 1.6. To measure this twisting, we use the notion of holonomy.

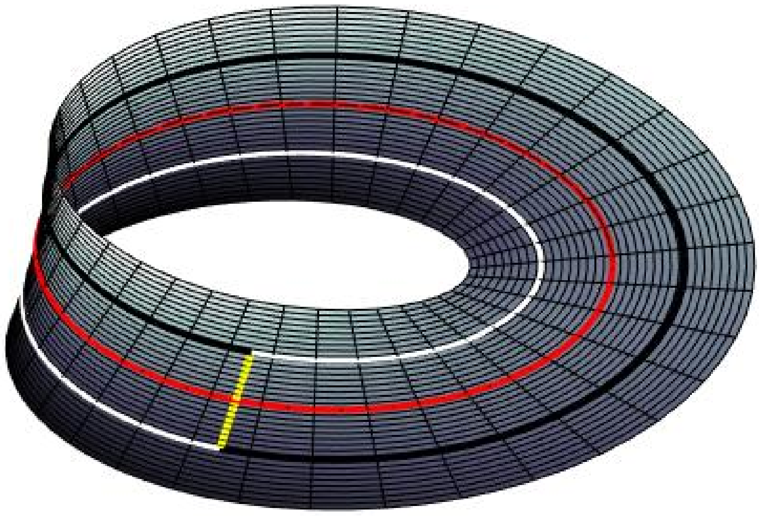


Figure 1.6: The twisting behaviour of the foliation ($T$: yellow)

Before continuing, we give a nice corollary of proposition 1.2.

**Proposition 1.3.** ([CN], pp. 51) Let $(M, \mathcal{F})$ be a foliated manifold, and $F$ a closed leaf of $\mathcal{F}$. Then $F$ is an embedded submanifold.

*Proof.* Let $(U, \phi)$ be a foliation chart of $\mathcal{F}$ such that $U \cap F \neq \emptyset$. Let $T$ be a transverse section whose closure is contained in $U$ and intersects $F$. There are only countably many plaques in $F \cap U$ in this chart, by definition of a foliation. Hence, $\overline{T} \cap F$ is also countable. Since $F$ is closed, this implies that $\overline{F \cap T}$ is countable. We claim that this implies that $F \cap \overline{T}$ is discrete. Indeed, suppose that it is not discrete. Then $F \cap \overline{T}$ has at least one point which is not isolated. We claim that this can only hold if $F \cap \overline{T}$ is perfect, in the sense no point is isolated. First, notice that $F \cap \overline{T}$ cannot have non-empty interior, since it must be countable. Suppose that there exists a point $p$ in $F \cap \overline{T}$ which is not isolated. Let $q$ be any other point in $F \cap \overline{T}$. By transverse uniformity, we know that any transversal section at $q$ can be diffeomorphically mapped to a transversal section contained at $p$ respecting the intersection with plaques. Since $p$ is not isolated, the transversal section will intersect another point $p'$ in $F \cap \overline{T}$. Thus, the transversal section at $q$ will intersect a point $q' \in F \cap \overline{T}$. Since we can shrink the transversal section as much as we want, this implies that $q$ is not isolated. Hence, $F \cap \overline{T}$ is indeed perfect, with empty interior. It is a result that a non-empty perfect set is necessarily uncountable. Hence, we conclude that $F \cap \overline{T}$ is discrete. It is easy to see that this condition implies that we can find a chart adapted to $F$, meaning that $F$ is a submanifold. □

## 1.2 Holonomy

### 1.2.1 Introduction and definition

Motivated by the proof of proposition 1.2, we shall define the notion of holonomy. In this section, we will follow the approach of I. Moerdijk's book *Introduction to foliations and Lie groupoids*, see ([MM], pp.20). To relax some of the notation, we introduce the following.

**Definition 1.6.** Let $(M, \mathcal{F})$ be a foliated manifold, and $F$ a leaf of $\mathcal{F}$. Then we say a path $\alpha : I \to M$ is a leafwise path if the image of $\alpha$ is contained in a single leaf. If the leaf $F$ is specified, we sometimes say $\alpha$ is an $F-$contained path.

**Remark.** The notation $F-$contained path is not standard.

**Notation 1.1.** If $\alpha : I \to M$ is a path from $x$ to $y$, we sometimes write $\alpha : x \mapsto y$.

We now fix a foliated manifold $(M, \mathcal{F})$ and a leaf $F$ on this foliation. We fix two points $x, y \in F$. Consider two transversal sections $S, T$ at $x$ and $y$ respectively. We will associate to any $F-$contained path $\alpha : x \mapsto y$ a germ of a diffeomorphism $\text{hol}(\alpha) = \text{hol}^{S,T}(\alpha) : (T, x) \to (S, y)$, called the holonomy of the path $\alpha$. Intuitively, these diffeomorphisms reflect what the behaviour of nearby leaves is, as observed by an inhabitant on the leaf $F$ as it travels along $\alpha$. Cover the image of $\alpha$ with a sequence of foliation charts. Then, the proof of proposition 1.2 yields us the desired diffeomorphism. We now show several small results regarding this construction.

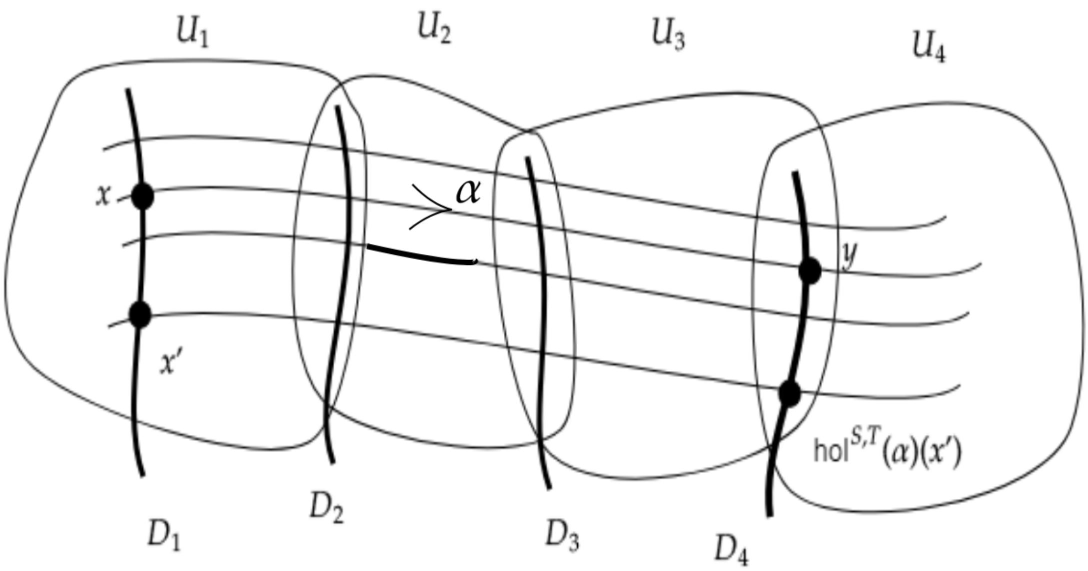



Figure 1.7: The holonomy of a path $\alpha$

1. Is this construction well-defined?
   For this, we need to check if the germ of the diffeomorphism depends on the choice of covering of foliation charts. The construction of the diffeomorphism was done locally, mapping points from one section to another by identifying points on the same plaque. Since plaques are independent of chosen chart, as shown by proposition 1.1, the germ of the diffeomorphism is independent of chosen covering.

2. Is this construction 'stable'?
   This means that a small perturbation of a given $F-$contained path $\alpha$ to another $F-$contained path $\beta$ gives the same germ of diffeomorphism. This is indeed the case: if the perturbation is small, then $\alpha$ and $\beta$ can be covered by the same sequence of foliation charts, hence giving the same diffeomorphism. In fact, suppose that $\alpha$ and $\beta$ are homotopic (as $F-$contained paths). Then since the construction is stable under perturbations, it will also be stable under $F-$contained path homotopies, since a homotopy can be seen as a way to go from $\alpha$ to $\beta$ by continuously perturbing.
   This drastically decreases the amount of possible (germs of) diffeomorphisms obtained via associating to each path $\alpha$ its holonomy $\mathrm{hol}^{S,T}$.

3. Concatenation of paths
   By our construction, it is quite easy to see that concatenation of paths coincides with the composition of germs of diffeomorphisms.

4. Choice of transversal section
   It is an easy observation that any two transversal sections $T, T'$ of $\mathcal{F}$ at $x$ are locally diffeomorphic. The germ of this diffeomorphism is given by $\mathrm{hol}(e_x)^{T,T'}$, where $e_x$ is the constant path at $x$. The dependence of choice of transversals is hence given by

$$\mathrm{hol}^{S',T'}(\alpha) = \mathrm{hol}^{S',S}(e_y) \circ \mathrm{hol}^{S,T}(\alpha) \circ \mathrm{hol}^{T,T'}(e_x).$$

Above list can be summarised by following result.

**Proposition 1.4.** ([MM], pp. 23) Given a foliated manifold $(M, \mathcal{F})$, a leaf $F$ of $\mathcal{F}$, a point $x \in F$ and a transversal section $T$ at $x$, we have a well-defined group homomorphism

$$\mathrm{hol}^{T,T} = \mathrm{hol}^T : \pi_1(F, x) \to \mathrm{Diff}_x(T),$$

where $\mathrm{Diff}_x(T)$ is the group of germs of diffeomorphisms at $x$, with $f(x) = x$ for every such germ. Since evidently $\mathrm{Diff}_x(T) \cong \mathrm{Diff}_0(\mathbb{R}^q)$, we can also write a homomorphism of groups

$$\mathrm{hol} : \pi_1(F, x) \to \mathrm{Diff}_0(\mathbb{R}^q),$$

called the holonomy homomorphism of F. This map is determined uniquely up to conjugation in $\mathrm{Diff}_0(\mathbb{R}^q)$. We write $\mathrm{Hol}(F, x) = \mathrm{hol}(\pi_1(F, x))$.

**Remark.** By transversal uniformity, the holonomy group of $F$ is independent of choice of base point, up to a conjugation in $\mathrm{Diff}_0(\mathbb{R}^q)$.

**Remark.** We can endow the space of leafwise paths with an equivalence relation via $\alpha_1 \sim \alpha_2$ if and only if $\mathrm{hol}(\alpha_1\overline{\alpha_2})$ =Id, where $\overline{\alpha_2}$ is the reverse of $\alpha_2$. One can easily define this equivalence relation to homotopy classes of paths. The equivalence class of $\alpha$, sometimes denoted $[\alpha]$, is called the holonomy class of $\alpha$.

### 1.2.2 Stability

Let $(M, \mathcal{F})$ be a foliated manifold, with the codimension of $\mathcal{F}$ being $q = n - \dim \mathcal{F}$. Fix some leaf $F$, and a base point $x_0 \in F$. Since hol is a group homomorphism from $\pi_1(F, x_0)$ to $\mathrm{hol}(\pi_1(F, x_0)) = H$, the kernel $K$ of hol is a normal subgroup of the fundamental group. Consider $\tilde{F}$, the covering space of $F$ which corresponds to $K$. Since $\pi_1(F, x_0)/K \cong H$, it follows that $H$ acts freely on $F$ and that $\tilde{F}/H \cong F$. We are interested in using this covering space to discuss the foliation near the leaf $F$. We first show following lemma.

**Lemma 1.2.** Let $F$ be a compact leaf with finite holonomy group. Then $\tilde{F}$ is compact.

*Proof.* Let $\{U_i\}$ be any open cover of $\tilde{F}$. Let $\{V_k\}$ be a covering of $F$ by evenly covered open sets. Denote by $\{S_l^k\}$ the slices obtained from $\{V_k\}$. For each $k$, there are $|H|$ amount of slices, which is finite. For a slice $\{S_l^k\}$, we have an open cover $\{U_i \cap S_l^k\}$. Since the projection map is a local homeomorphism on these slices, this induces an open covering of $V_k$. This induces a finer covering of $F$, which by compactness has a finite subcovering. This in turns defines a finite subcover of the slice $S_l^k$. This can be done for every $k$ and $l$. Since both are finite index sets, we are done. □

Remaining in the setting where $F$ is some compact leaf with finite holonomy group, we can define a special transversal section $T$ as follows. Notice that an element of $H$ is the germ of a locally defined diffeomorphism, defined on some transversal section. Since we have only finitely many elements in $H$, we can represent all elements by their holonomy diffeomorphism by shrinking $T$. This allows us to formulate the local Reeb stability theorem.

**Theorem 1.1** (Reeb Stability,[Rb])**.** Let $F$ be a compact leaf whose associated holonomy group is finite, $T$ a transversal section as above. Then there is a neighbourhood V of $F$ that is a union of leaves of $\mathcal{F}$ and a diffeomorphism of foliated manifolds

$$\tilde{F} \times_H T \to V$$

which identifies the foliation $\mathcal{F}$ on $V$ with the foliation on $\tilde{L} \times_H T$ seen in example 1.10.

In other words, in this setting the nearby leaves look like covering spaces of $F$, whose fibers are finite, giving us a normal form. It depends only on the leaf $F$ and the holonomy of this leaf. Combining this with lemma 1.2, we find that for compact leafs $F$ whose associated holonomy group is finite, nearby leaves must also be compact.

We call the space $\tilde{F}$ the <u>holonomy cover</u> of $F$, since not only is $\tilde{F}$ a covering space of $F$, each element of $\tilde{F}$ can be seen as a holonomy class $[\alpha]$, where $\alpha$ is an $F-$contained path from $x_0$ to $y$.

**Remark.** A special case of above theorem is when $H = 1$. This tells us that the foliation near the leaves looks like the product of the leaf and a small transversal.

In a certain sense, we can look at the holonomy cover of $F$ (relative to some base point $x_0$) as the set of triples

$$\tilde{F} = \{(x_0, y, [\alpha]) | \alpha \text{ is an F-contained path from } x_0 \text{ to } y\}.$$

As shown above, these covering spaces encode a lot of information about the foliation. However, it is evident that in general we cannot recover any global information from these objects. In the following chapter, we discuss how we can fit these spaces together in a so-called groupoid which is strong enough to discuss the foliation globally whilst still retaining a lot of favourable properties.

**Example 1.12.** As an example, we show how the holonomy cover of the central leaf $F$ of the Möbius foliation looks like. Recall that the holonomy group of this leaf corresponded to $\mathbb{Z}/2\mathbb{Z}$. The holonomy cover is the two-sheeted covering, as seen in figure 1.8.

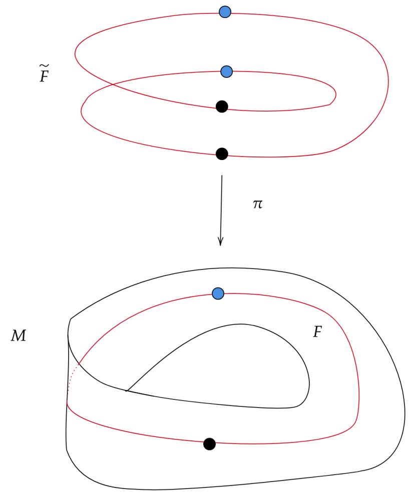



Figure 1.8: The holonomy cover of the central leaf

# Chapter 2

# Lie groupoids and Lie algebroids

In this chapter, we discuss Lie groupoids and Lie algebroids. These generalisations of Lie groups and Lie algebras will be the cornerstones in our study of holonomy. In particular, we will see that regular foliations are special examples of Lie algebroids. These specific Lie algebroids can always be integrated to a Lie groupoid, similarly as a Lie group integrates a Lie algebra. The holonomy groupoid, which gathers all holonomy in a single algebraic and smooth object, will be an example of a Lie groupoid that integrates the foliation. Good references for this chapter are [MM],[Gd], [McK],[WdS] and [Mein].

## 2.1 Groupoids

### 2.1.1 Definition

As an introduction, we start with an introductory example. Recall that at each base point $x$ of a topological space X, we have a group $\pi_1(X,x)$ associated to it. In the case of a path-connected space, this group is sufficient to determine the fundamental group at every point in the space. However, in general this is not the case. For a space with many path-connected components, the fundamental groupoid can be of use.

**Definition 2.1.** [Gd] The fundamental groupoid $\Pi(X)$ of a topological space $X$ is the set

$$\{(x,y,[\alpha]) : x,y \in X, \alpha : x \to y\},$$

where the equivalence class in question comes from path homotopy (i.e homotopies of paths fixing the endpoints), endowed with the partially defined operation

$$\{(x,y,[\alpha])\} \circ \{(y,z,[\beta])\} = \{(x,z,[\alpha * \beta]\}.$$

With this definition in mind, the following definition makes sense.

**Definition 2.2.** [MM] A groupoid $G$ consists of the following data:

1. A set of objects $G_0$.

2. A set of arrows $G_1$, also called morphisms.

3. Two maps $s,t : G_1 \to G_0$ called the source and target map.

4. An associative, partially defined multiplication on $G_1$, denoted $\mu(h, g)$ or just $hg$, which is only defined when $s(h) = t(g)$.

Furthermore, every object $x$ has an associated unit $1_x$ satisfying the property $1_{t(g)}g = g = g1_{s(g)}$. Furthermore, every arrow $g$ has an associated inverse arrow $g^{-1}$ for which $s(g) = t(g^{-1})$, $t(g) = s(g^{-1})$ and $gg^{-1} = 1_{s(g)}$, $g^{-1}g = 1_{t(g)}$.

**Remark.** From associativity of the multiplication, it follows that $t(hg) = t(h)$ and $s(hg) = s(g)$.

**Notation 2.1.** There are several sets associated to a groupoid.

$$G(x, y) = \{g \in G_1 | t(g) = y, s(g) = x\}$$

is the set of arrows $g : x \mapsto y$. Notice that $G(x, y) = s^{-1}(x) \cap t^{-1}(y)$. We can also define

$$G(x, -) = s^{-1}(x), \quad G(-, y) = t^{-1}(y).$$

Furthermore, the isotropy group $G_x$ is defined $G_x = G(x, x)$.

**Example 2.1.** A groupoid with just one object is a group. Indeed, in this case, the multiplication is defined for all morphisms, and one notices that in this case the morphisms form a group under this multiplication. Often one says that a groupoid is a 'many-objects' generalisation of a group.

**Example 2.2.** The isotropy groups of the fundamental groupoid $\Pi(M)$ are the fundamental groups of the space. This is easily seen: the isotropy groups consist of the paths ending at its starting point, i.e loops. This shows that the fundamental groupoid is a generalisation of the usual notion of the fundamental group. In particular, whilst the fundamental group based at some point $x$ captures only the information of the path connected component of $x$, the fundamental groupoid encodes the information for all connected components at once.

Before continuing, we give a way to intuitively think about groupoids. The idea is to identify the objects $G_0$ with the unit space $1_G$. Then, we consider the fibers of the source and tangent map, see figure 2.1.

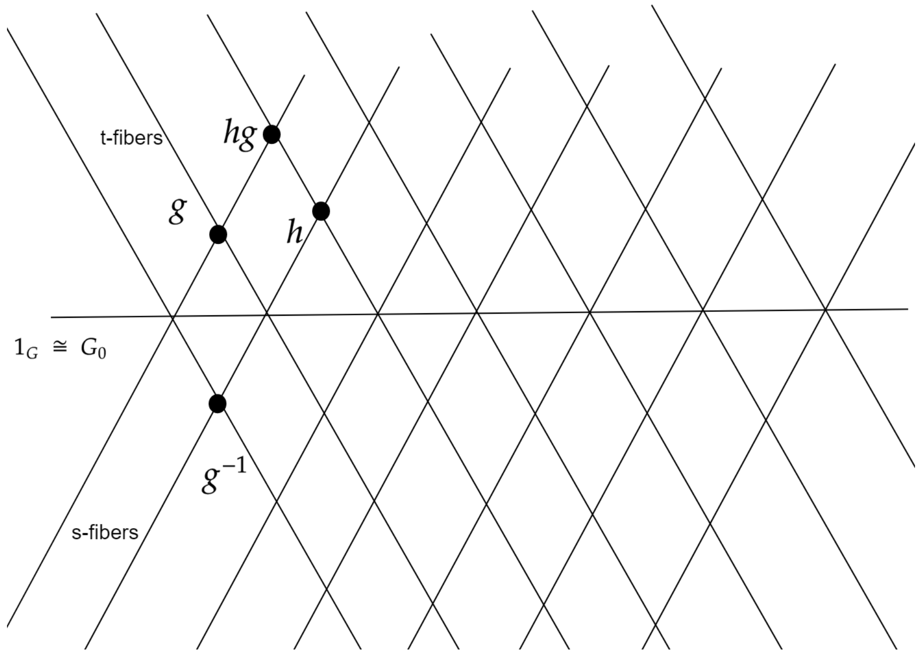



Figure 2.1: A graphical representation of a groupoid

### 2.1.2 Lie groupoids

At the moment, we have a purely algebraic structure. We now bring it in a smooth setting.

**Definition 2.3.** [McK] A Lie groupoid $G$ is a groupoid such that

1. The base manifold $G_0$ is a smooth Hausdorff manifold.
2. The arrow space $G_1$ is a smooth manifold, but might not be Hausdorff.
3. The multiplication, inversion, source and unit map (which maps $x$ to $1_x$) are all smooth.
4. The source map (and therefore the target map) are surjective submersions.

**Remark.** Notice that the unit map is a smooth section of the submersion $s$. Hence, $s \circ u$ is the identity on $G_0$, which implies $u$ is an immersion. Furthermore, the unit map is a homeomorphism onto its image. Hence, its image is an embedded submanifold of $G_1$, and we denote it by $1_G$. For this, and many of the following results, it is useful to keep the graphical representation (figure 2.1) in mind.

As always, we first consider some examples.

**Example 2.3.** Given any manifold $M$, we can regard it as a Lie groupoid. Its space of arrows is $1_M$, and the source and target map are the identifications $1_x \mapsto x$.

**Example 2.4.** Let $M$ be a manifold, then another trivial example is the *pair groupoid*, denoted $M \times M$. The elements are pairs $(x, y)$, and the partial multiplication is defined fully by the equation

$$(x, y) * (y, z) = (x, z).$$

The source and target map are the first, respectively the second projection.

**Example 2.5.** [MM] A more interesting example comes from the theory of Lie groups. Suppose $G \times M \to M$ is a smooth Lie group action. The product manifold $G \times M$ can be seen as a Lie groupoid on $M$ as follows. The source map is the second projection, whilst the target map is $t(g, x) = gx$. The unit map is $x \mapsto (e, x)$ with $e$ the neutral element of $G$. The partial multiplication is defined $(g, hx)(h, x) = (gh, x)$.
This groupoid is denoted $G \rightrightarrows M$, and is called the action groupoid of the action. Notice that it encodes the information of the action.

**Example 2.6.** The fundamental groupoid is a Lie groupoid.[Gd]

**Example 2.7.** [MM] Given a submersion $p : N \to M$, we may define the Lie groupoid $\mathrm{Ker}(p)$ with arrows $\mathrm{Ker}(p)_1 = N \times_M N$ and objects $N$. The (fiberwise) multiplication is analogous to the multiplication of the pair groupoid, and in fact is a subgroupoid of $\mathrm{Pair}(M)$.

In the remainder of this section, we look at some basic results about Lie groupoids. These results, including the proofs, come from [MM].

**Proposition 2.1.** [MM] The fibres $G(x, -)$ and $G(-, x)$ are embedded submanifolds. The natural action of $G_x$ on $G(x, -)$ is free and transitive along the fibres of $t|_{G(x,-)} = t_x$.

*Proof.* The proposition consists of two parts. The first part follows from the regular value theorem. For the second part, the action is given by a right multiplication. To show that the action is transitive along $t_x$−fibres, choose any $\alpha : x \to y$ and $\beta : x \to y$. Consider the arrow $\alpha^{-1}\beta : x \to x$. By definition, this lies in the isotropy group. By construction, right multiplication of $\alpha$ with this arrow gives $\beta$, implying the action is indeed transitive. For the action to be free, suppose $\alpha\beta = \alpha$ with $\alpha$ any arrow in a fibre of $t_x$ and $\beta$ an element of the isotropy group. Then evidently, $\alpha^{-1}\alpha\beta = 1_x$, which implies $\beta = 1_x$. □

**Proposition 2.2.** [MM] Let $G$ be a Lie groupoid, and let $x, y \in G_0$. Then $G(x, y)$ is an embedded submanifold of $G$.

*Proof.* Consider the vector spaces $E_g = \ker(ds)_g \cap \ker(dt)_g$ for all $g \in G$. It is clear from the regular value theorem that, at each point $g$ of $G(x, y)$, the tangent space should be $E_g$. To show that $G(x, y)$ is indeed a submanifold, we first show that $E|_{G(x,-)}$ is a distribution on $G(x, -)$. For this, we have to show that the dimension of $E_g$ is constant for all $g \in G(x, -)$ and that $E_g$ varies smoothly on $G(x, -)$. We fix an element $g \in G(x, -)$. Associated to this is the diffeomorphism $L_g : G(-, x) \to G(-, t(g))$, which is the left multiplication by $g$. We claim that $(dL_g)_{1_x}(E_{1_x}) = E_g$. Notice that this map is well-defined: let $h \in G(-, x)$, then $E_h = \ker(dt)_h \cap \ker(ds)_h \subset \ker(dt)_h = T_h(G(-, x))$. Since $L_g$ is constant on the fibers of $s$, we furthermore find $s \circ L_g = s|_{G(-,x)}$. Since $L_g$ is a diffeomorphism, we find that $(dL_g)_{1_x}(E_{1_x}) = E_g$, showing that they are of the same dimension. However, this also allows us to show that $E_g$ varies smoothly for $g \in G(-, x)$: it suffices to choose a local basis $e_1, ..., e_n$ of $E_{1_x}$ and extend it to a global frame via the diffeomorphisms $dL_g$. Thus, $E_g$ is indeed a subbundle of $G(x, -)$. Notice that this subbundle is exactly the kernel of $dt|_{G(x,-)}$. By what we have just shown, the rank of this map is constant. Hence, the fibre $t_x^{-1}(y)$ is an embedded submanifold. But this is exactly $G(x, y)$, showing the desired result. □

**Remark.** Notice that this also implies that $G_x$, the isotropy group, is a Lie group.

## 2.2 Lie Algebroids

### 2.2.1 Introduction

A Lie algebra is seen as an infinitesimal approximation of a Lie group. In this section, we want to look at the analogous structure, called the Lie algebroid, of a Lie groupoid. Later, we shall see that this infinitesimal approximation is a natural extension of viewing an integrable subbundle as the infinitesimal approximation of a foliation. In this section, we use the same approach as [MM].

In the construction of a Lie algebra from a Lie group, we crucially needed the fact that for each $g$, right translation is a diffeomorphism of $G$. However, the action of an arrow $h : x \mapsto y$ is not everywhere defined. What we do get, however, are diffeomorphisms $R_h : s^{-1}(y) \to s^{-1}(x)$, coming from the right multiplication. In this case, $dR_h$ is an isomorphism from $ker(ds)_y$ to $\ker(ds)_x$. In general, this procedure gives rise to a right action of the Lie groupoid $G$ on the vector bundle $\ker(ds) = T^s(G_1)$. Indeed, given any $\xi \in (T^s_g(G_1))$ and any $h \in G_1$ composable with $g$ (i.e $t(h) = s(g)$), we can define

$$\mu(\xi, h) = dR_h(\xi) \in T^s_{gh}(G_1).$$

Notice that $dR_h(\xi) \in T^s_{gh}(G_1)$ since the image of $R_h$ is contained in the fiber $s^{-1}(x)$. Now that we know what vectors we want to work with, we can simply define a right invariant vector field to be a section of the bundle $\ker(ds)$ (seen as a subbundle of the tangent bundle) such that for all composable $g, h$:

$$\mu(X(g), h) = X(gh).$$

We denote this space by $\mathfrak{X}^s_{inv}(G_1)$. We denote by $\mathfrak{X}^s(G_1)$ the vector fields tangent to the fibers. Notice that $\mathfrak{X}^s_{inv}(G_1) \subset \mathfrak{X}^s(G_1)$.We look at some properties of above construction.

**Proposition 2.3.** [MM] For a Lie groupoid $G$, we have

1. $\mathfrak{X}^s(G_1)$ and $\mathfrak{X}^s_{inv}(G_1)$ are Lie subalgebras of $\mathfrak{X}(G_1)$.
2. The right-invariant vector fields $\mathfrak{X}^s_{inv}(G_1)$ are $t-$projectable. The map $dt : \mathfrak{X}^s_{inv}(G_1) \to \mathfrak{X}(G_0)$ is a homomorphism of Lie algebras.

**Remark.** Recall that a vector field $x$ is $t-$projectable if and only if $dt(X)$ is constant on the fibers of $t$.

*Proof.* Consider the vector bundle $T^s(G_1)$, which consists of the vectors that are tangent to the source-fibers $s^{-1}(x)$ for all $x$. Since these fibers are closed submanifolds, these vector fields are closed under the Lie bracket. Thus $\mathfrak{X}^s(G_1)$ is a Lie subalgebra of $\mathfrak{X}(G_1)$. Suppose now that $g, h$ are two composable arrows. Choose two right-invariant vector fields $X, Y \in \mathfrak{X}^s_{inv}(G_1)$. Recall that the Lie bracket of $R_h-$related vector fields are $R_h-$related. Since, by definition, a right-invariant vector field is $R_h-$related to itself, it follows that $dR_h([X, Y])_g = [dR_h(X), dR_h(Y)]_{gh}$. Since $X, Y$ were by assumption right-invariant, the result follows. This shows item (1), let us now consider item (2). Let $h \in G(x, y)$ be any arrow, then

$$dt(X_h) = dt(dR_h(X_{1_x})) = dt(X_{1_x}).$$

Here, we used the fact that $(t \circ R_h)(g) = t(g)$, i.e $t$ is invariant under the right action. The final statement now easily follows. □

Recall that in the case of Lie groups, we had a bijection from the right-invariant vector fields and the tangent vectors at the unit. This is done via $X_g = (R_g)_*(X_e)$. Analogously, we can fully determine the values of a right-invariant vector field in the case of Lie groupoids if we know the tangent vectors on the set of units, i.e $\{1_x | x \in G_0\}$. Hence, we are interested in a vector bundle on $G_0$ (which is diffeomorphic to the set of units) that assigns to each point a vector in $T^s(G_1)$. This is given by the pull-back of the following diagram:

$$\begin{array}{ccc} \mathfrak{g} & \dashrightarrow & T^s(G_1) \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ G_0 & \xrightarrow{x \mapsto 1_x} & G_1 \end{array}$$

We regard $\mathfrak{g}$ as the vector bundle over $M$ whose fiber over a point $x$ is $T^s_{1_x}(G_1)$. Then, we can identify $\Gamma(\mathfrak{g})$ with $\mathfrak{X}^s_{inv}(G_1)$, i.e they are isomorphic. Pulling back the Lie algebra structure on $\mathfrak{X}^s_{inv}(G_1)$, this is even a Lie algebra isomorphism.

We can naturally look at the derivative of the target map as a map from $\mathfrak{g}$ to $T(G_0)$. This gives us a vector bundle homomorphism $\# : \mathfrak{g} \to T(G_0)$. At the level of sections, this gives us a homomorphism of Lie algebras (again denoted by $\#$) $\# : \Gamma(\mathfrak{g}) \to \mathfrak{X}(G_0)$. By construction, this corresponds to the projection of $\mathfrak{X}^s_{inv}(G)$ as above.

The vector bundle $\mathfrak{g}$ over $G_0$ we just constructed is called the Lie algebroid of the Lie groupoid $G$. For a graphical representation, see figure 2.2. This is a special case of a more general definition of a Lie algebroid, which is the content of next section. These graphical representations are based on the ones found in ([WdS], section 7).

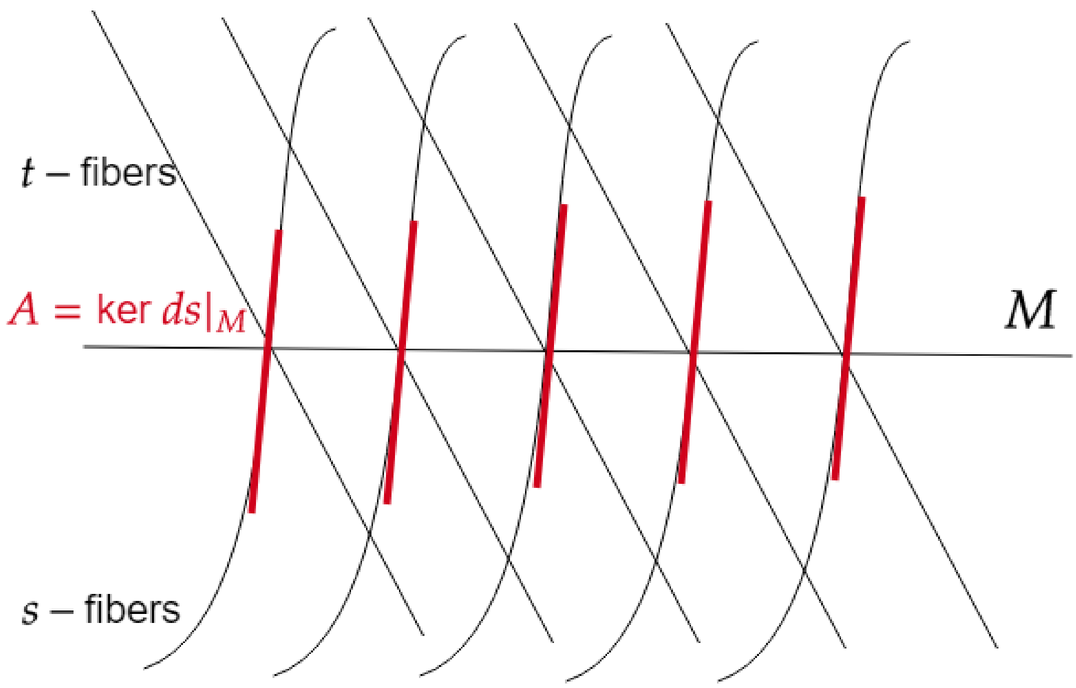



Figure 2.2: A graphical representation of the Lie algebroid associated to a Lie groupoid

### 2.2.2 Definition and examples

We start by giving the definition of a Lie algebroid.

**Definition 2.4.** [MM] Let $M$ be a smooth manifold. A Lie algebroid over $M$ is a vector bundle $\pi : A \to M$, together with a vector bundle map (called the anchor map) $\# : A \to TM$ and a real Lie algebra structure $[,]$ on its space of sections $\Gamma(A)$ such that

1. The induced map of sections $\# : \Gamma(A) \to \mathfrak{X}(M)$ is a Lie algebra homomorphism.
2. The Leibniz identity
$$[X, fY] = f[X,Y] + \#(X)(f)(Y)$$
holds for any $X, Y \in \Gamma(A)$ and $f \in C^\infty(M)$.

We now give some examples. These examples can be found in [MM].

**Example 2.8.** Recall that a Lie group could be seen as a Lie groupoid over a single object. It would therefore be natural for the Lie algebroid of this Lie groupoid to coincide with the Lie algebra of the Lie group. This is indeed the case, notice that $A$ is a vector bundle over one point (which can naturally be seen as a vector space) together with a Lie algebra on its space of sections. A section on this Lie algebroid corresponds to assigning a vector to $1_x$, which corresponds to the unit of the Lie group. Thus, the space of sections coincide with the tangent space of $G$ at the unit, which is exactly the Lie algebra of $G$. Indeed, notice that the (only) $s-$fibre is the whole space, and hence all vectors are tangent: $T^s(G_1) = TG_1$.

**Example 2.9.** Vector bundles can be seen as a special case of Lie algebroids, with trivial Lie algebra structure on the sections and zero anchor.

**Example 2.10.** Just like in the case of Lie groupoids, a manifold can be made into a Lie algebroid, by considering the bundle $\{0\} \times M$ with trivial anchor map. This Lie algebroid is integrated by the Lie groupoid we saw in example 2.1. This Lie algebroid corresponds to the unit Lie groupoid.

**Example 2.11.** A foliation $\mathcal{F}$ on a smooth manifold $M$ is a special case of a Lie algebroid. The vector bundle $A$ is given by the distribution, the anchor map is the inclusion. Notice that in this case, since the distribution is involutive, it inherits naturally the Lie algebra structure of $\mathfrak{X}(M)$. In the opposite direction, suppose one has a Lie algebroid $A$ with injective anchor map $\# : A \to TM$. Then, one can identify $A$ with $\#(A)$, which by definition of a Lie algebroid is a subbundle of $TM$. Since $\#$ is a Lie algebra homomorphism, this vector bundle is furthermore involutive. Therefore, the image of a Lie algebroid with injective anchor map gives rise to a foliation.

Notice that in the first three examples, the Lie algebroids given came from Lie groupoids. A Lie algebroid that comes from a Lie groupoid is called integrable. It is not true that every Lie algebroid comes from a Lie groupoid. For an example of this, see section 6.3.2. A natural question is whether a foliation which by above can be viewed as a Lie algebroid is integrable. The answer to this question is positive. In the next section, we will construct two (in general different) Lie groupoids that integrate such foliations $\mathcal{F}$.

**Example 2.12** ([WdS],[MM]). Another nice example comes from the theory of Lie algebra actions. Recall that a Lie algebra action of a Lie algebra $\mathfrak{g}$ on $M$ is a Lie algebra homomorphism

$$\gamma : \mathfrak{g} \to \mathfrak{X}(M).$$

We can endow the trivial bundle $\mathfrak{g} \times M$ with a Lie algebroid structure as follows. The anchor map is defined

$$\#(\alpha, p) = \gamma(\alpha)_p.$$

The Lie bracket is defined

$$[u, v](x) = [u(x), v(x)] + (\gamma(u(x))(v))(x) - (\gamma(v(x))(u))(x),$$

where $u, v \in C^\infty(M, \mathfrak{g} \times M)$ and $x \in M$. This Lie algebroid is called the transformation Lie algebroid of the action of $\mathfrak{g}$ on $M$. We denote this Lie algebroid by $\mathfrak{g} \ltimes M$. Recall that any Lie group action induced a Lie algebra action. In this case, one can show that the Lie algebroid $\mathfrak{g} \times M$ associated to this infinitesimal action can be integrated by the action Lie groupoid $G \rightrightarrows M$. In fact, it was shown by Dazord (see [Daz]) that any transformation Lie algebroid is integrable.

## 2.3 The holonomy and monodromy groupoids

In this section, we construct two groupoids which correspond to the foliation. Recall that in the introduction, we briefly mentioned the fact that the equivalence relation $\mathcal{R}$ induced by a foliation need not be smooth. What we mean by this is that

$$\mathcal{R} = \{(x, y)|x, y \text{ lie on the same leaf}\} \subset M \times M$$

is not an immersed Lie subgroupoid of $M \times M$ (recall that $M \times M$ could be seen as a Lie groupoid, called the pair groupoid $\mathrm{Pair}(M)$).

The question is now: can we find a Lie subgroupoid of $\mathrm{Pair}(M)$ whose orbits induce the foliation? The answer is yes, but in fact we know more: the Lie subgroupoid we shall construct will be the *smallest* (see [MM2]) Lie groupoid inducing the foliation on $M$. This is the holonomy groupoid. We will also discuss its homotopic sibling, called the monodromy groupoid. The objects of the groupoids will be points on the manifold, and the arrows will consist of leafwise paths.

**Definition 2.5.** [MM] Let $(M, \mathcal{F})$ be a foliated manifold. Define the groupoid $\mathrm{Mon}(M, \mathcal{F})$ to be the groupoid over $M$ whose arrow space is

$$\mathrm{Mon}(M, \mathcal{F})(x, y) = \{[\gamma]|\gamma : x \mapsto y \text{ a leafwise path}\}.$$

Here, the equivalence class is given by path-homotopy (i.e fixing endpoints) with respect to leafwise paths. Similarly, we define the groupoid $\mathrm{Hol}(M, \mathcal{F})$ to be the groupoid over $M$ whose arrow space

$$\mathrm{Hol}(M, \mathcal{F})(x, y) = \{[\gamma]|\gamma : x \mapsto y \text{ a leafwise path}\},$$

but here the equivalence relation is given by holonomy. The partially defined multiplication is the one coming from concatenation of paths.

We have to show that these groupoids can be endowed with a smooth structure such that they are Lie groupoids. The following proposition and proof can be found in ([MM], prop. 5.6).

**Proposition 2.4.** [MM] The holonomy and monodromy groupoids are Lie groupoids.

*Proof.* We will prove this for the monodromy groupoid. The construction for the holonomy groupoid is completely analogous. We are going to construct a smooth atlas for the arrow space. Given any point $(x, y, \alpha)$, we need to find a neighbourhood $W$ at $(x, y, \alpha)$ and a diffeomorphism $f : W \to \mathbb{R}^N$ in such a way that the smooth maps are compatible as local charts. Furthermore, this construction needs to be independent of chosen representative at the level of homotopy. Let $(\phi, U)$ and $(\psi, V)$ be foliation charts around $x$ and $y$ respectively. By shrinking $U$ if necessary, we can assume that the images of these charts are of the form $A \times C$ and $B \times D$, where $A$ and $B$ are open subsets of $\mathbb{R}^k$ and $C, D$ are open subsets of $\mathbb{R}^{n-k}$. Here, $k$ is taken to be the dimension of the foliation. We shrink the domains of the charts if necessary to obtain that $A, B, C$ and $D$ are all connected and simply connected. We write $x = \phi(a, c)$ and $y = \psi(b, d)$. Notice that then, $S = \phi^{-1}(\{a\} \times C)$ and $T = \phi^{-1}(\{b\} \times D)$ are transversal sections through $x$ and $y$ respectively. Let $\gamma$ be a representative of the $F-$contained path homotopy class $\alpha$. This induces a holonomy transformation $\mathrm{hol}^{S,T}(\alpha)$, which can be realised as a diffeomorphism from $S$ to $T$ by shrinking the transversal sections if necessary. By the construction of the holonomy transformation, one can associate to each leaf $F'$ intersecting $S$ an $F'-$contained path which is unique up to homotopy. Indeed, it suffices to consider any path in the plaque connecting the intersection points, and concatenating these paths together. Hence, we can define a map $H : [0,1] \times S \to M$ such that $H|_{\{0\}\times S} = Id_S$ and $H|_{\{1\}\times S} = \mathrm{hol}^{S,T}(\gamma)$ and $H(t,z)$ is a leafwise path for each $z$. Intuitively, this map slides $S$ along the leaves towards $T$, following $\gamma$. This allows us to define a map

$$f : A \times B \times C \to \mathrm{Mon}(M,\mathcal{F})_1$$

as follows. For any triple $(a',b',c')$, we consider a leafwise path $\nu_1$ from $\phi^{-1}(a',c')$ to $\phi^{-1}(a,c')$, thus bringing the points on every plaque to the intersection of the plaque with $S$. Then, we consider any path $\nu_2$ from $\phi^{-1}(a,c')$ to $\psi^{-1}(b,d')$ described by $H(-,\phi^{-1}(a,c'))$. Finally, we let $\nu_3$ be any path from $\phi^{-1}(b,d')$ to $\phi^{-1}(b',d')$. By construction, each of these paths are leafwise paths. Hence, the homotopy class of their concatenation is an element of $\mathrm{Mon}(M,\mathcal{F})_1$. Notice that $f$ is injective, so $f$ is a bijection onto its image. We take the sets that are obtained this way as a basis of open sets for a topology on $\mathrm{Mon}(M,\mathcal{F})_1$. One can show that these type of maps $f$ form a smooth chart on $\mathrm{Mon}(M,\mathcal{F})_1$, which makes it into a Lie groupoid. □

**Remark.** Let us consider some of the associated sets of these Lie groupoids. First, given $x \in M$, notice that the isotropy group $\mathrm{Mon}(M,\mathcal{F})_x$ is nothing else than $\pi_1(F,x)$, with $F$ the leaf containing $x$. Analogously, we have $\mathrm{Hol}(M,\mathcal{F})_x = \mathrm{Hol}(L,x)$.
Furthermore, notice that $\mathrm{Mon}(M,\mathcal{F})(x,-)$ corresponds to the set of triples $(x,y,[\alpha])$, where $\alpha$ is an $F-$contained path from $x$ to $y$. This corresponds to the universal covering space of $F$, and thus the map sending sending a homotopy class of paths to its associated holonomy class restricts to the universal covering map. Analogously, the source fibre for the holonomy groupoid coincides with the holonomy covering of the leaf $F$, and the target map hence restricts to the holonomy covering projection.

**Proposition 2.5.** [MM] The orbits of the monodromy groupoid and the holonomy groupoid are exactly the leaves of $\mathcal{F}$.

*Proof.* Let us prove this result for the monodromy groupoid. The proof for the holonomy groupoid is analogous. Fix some point $x \in F$, where $F$ is a leaf of $\mathcal{F}$. Let $y$ be any other point on $F$. We have to show that there exists an element $\alpha \in \mathrm{Mon}(M,\mathcal{F})$ such that $s(\alpha) = x$ and $t(\alpha) = y$. This just follows from the fact that the leaves are path-connected spaces. Hence, $t(G(x,-)) = F$. □

**Proposition 2.6.** [MM] The isotropy groups of the monodromy and holonomy groupoids of $(M,\mathcal{F})$ are discrete groups.

*Proof.* This is immediate from the construction. □

**Proposition 2.7.** The Lie algebroids of the Lie groupoids $\mathrm{Hol}(M,\mathcal{F})$ and $\mathrm{Mon}(M,\mathcal{F})$ can both be identified with the Lie algebroid associated to the foliation.

*Proof.* For any given $x \in M$, the source fibre is the holonomy cover of the leaf $F$ containing $x$ with holonomy projection $t$. Thus, the anchor map $\#(\xi) = dt(\xi)$ maps $\mathfrak{g}_x$ bijectively onto the vectors tangent to the leaf $F$. Therefore, we can interpret the vector bundle $\mathfrak{g}$ as the distribution of our foliation, which is what we wanted to show. □

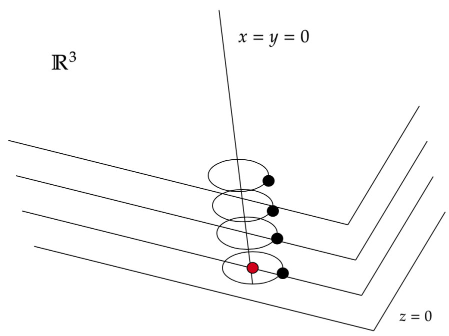
To finish this chapter, we look at some examples of holonomy and mondromy groupoids.

**Example 2.13.** Consider the foliation of the Möbius band $M$. This foliation came from the action of $S^1$ onto $M$ which wrapped $S^1$ around $M$ twice.
As one can expect, the holonomy groupoid in this case coincides with the action groupoid $S^1 \rightrightarrows M$.

**Example 2.14.** Consider the trivial foliation of a manifold with a single leaf $M$. Then we find

$$\mathrm{Mon}(M, \mathcal{F}) = \mathrm{Hol}(M, \mathcal{F}) = M \times M.$$

**Example 2.15.** Recall that in the definition of Lie groupoids, we did not require the space of arrows to be Hausdorff. This might seem surprising, but we now mention an example that showcases that these arise naturally in our context. Consider the punctured space $\mathbb{R}^3 \setminus \{(0,0,0)\}$. This can be foliated using horizontal planes, each one is diffeomorphic to each other (and in particular diffeomorphic to $\mathbb{R}^2$) except for the leaf at $z = 0$, which is diffeomorphic to $\mathbb{R}^2 \setminus (0,0)$. Consider the monodromy groupoid. At every leaf away from the central leaf, this is trivial since the spaces are homotopy equivalent to a point. Consider a sequence of loops $\gamma_n$ at height $z = 1/n$. The limit of this sequence is not unique, see figure 2.15. There, one sees that the loop around the $x = y = 0-$axis has as a limit a loop around the origin (and hence is not contractible), whilst the sequence of points has as limit the corresponding point on the $z = 0$-plane. This example gives us an example of where the monodromy groupoid fails to be Hausdorff.




Thus, for each foliated manifold $(M, \mathcal{F})$ we have found a Lie groupoid that 'integrates' the foliation. In the next chapters, we will extend this idea to a more general case.

# Chapter 3

# Singular Foliations

Whereas regular foliations can be seen as a decomposition of a manifold into leaves of the same dimension, a singular foliation may exhibit leaves of different dimensions. This new type of foliation is not pathological: they arise naturally in many different contexts. One example comes from the theory of Lie algebroids: recall that a Lie algebroid whose anchor was injective gave rise to regular foliations. Dropping this injectivity condition, one still gets a partition but this time the dimension of the leaves can change. This type of Lie algebroid pops up when studying group actions, but also in topics like Poisson geometry. In this chapter, we will talk about the basic theory surrounding singular foliations. Relevant sources are [DZ],[Gd],[W],[AS], [WdS], [AZ] and [AZ2].

## 3.1 Introduction

### 3.1.1 Definition and examples

Most of the following definitions and results can be found in [AS].

**Definition 3.1.** [AS] Let $M$ be a smooth manifold. A (singular) foliation $\mathcal{F}$ is a locally finitely generated $C^\infty(M)-$submodule of $C_c^\infty(M, TM)$, the compactly supported vector fields, that is stable under the Lie bracket.

**Remark.** Notice that, instead of working with subbundles of the tangent bundle (as we did in the case of regular foliations) we look at submodules of the sections of the tangent bundle.

We recall what it means for a submodule to be finitely generated.

**Definition 3.2.** [AS] A submodule $\mathcal{F} \subset C_c^\infty(U, TU)$ is said to be finitely generated if we can find $X_1, ..., X_n \in C^\infty(U, TU)$ such that $\mathcal{F} = C_c^\infty(U) \cdot X_1 + ... C_c^\infty(U) \cdot X_n$. A submodule $\mathcal{F} \subset C_c^\infty(M, TM)$ is said to be locally finitely generated if, for every $x \in M$, one can find a neighbourhood $U$ of $x$ such that $\{X|_U : X \in \mathcal{F}, \text{support}(X) \subset U\}$ is finitely generated.

**Remark.** From this point onward, when we talk about a foliation $\mathcal{F}$ we mean a singular foliation.

To measure how singular a foliation is near a point, we define the following.

**Definition 3.3.** [AS] Let $(M, \mathcal{F})$ be a foliated manifold. For any $x \in M$, one defines the maximal ideal

$$I_x = \{f \in C^\infty(M) | f(x) = 0\}.$$

One defines the tangent space of the leaf of $\mathcal{F}$ at $x$ by $F_x = \{ev_x(X) | X \in \mathcal{F}\}$, which is a subset of $T_xM$. The fibre of $\mathcal{F}$ at $x$ is defined $\mathcal{F}_x = \mathcal{F}/I_x\mathcal{F}$.

As shown in [AZ3] and [AS],we have a map $ev_x : \mathcal{F} \to T_xM$, which vanishes on $I_x\mathcal{F}$. Hence, this map drops down to a surjective morphism $e_x : \mathcal{F}_x \to F_x$. The kernel of $ev_x$, i.e all vector vector fields $X \in \mathcal{F}$ such that $X(x) = 0$, is a Lie subalgebra of $\mathcal{F}$. We denote this Lie subalgebra by $\mathcal{F}(x)$. Notice that $I_x\mathcal{F}$ is a Lie ideal of $\mathcal{F}(x)$, from which it follows that $\mathcal{F}(x)/I_x\mathcal{F}$ is a Lie algebra $\mathfrak{g}_x$, called the infinitesimal isotropy of $\mathcal{F}$ at $x$. Furthermore, we have following short exact sequence of vector spaces:

$$0 \longrightarrow \mathfrak{g}_x \longrightarrow \mathcal{F}_x \xrightarrow{ev_X} F_x \longrightarrow 0$$

We now show how these concepts measure how singular the foliation near a point is. The following results and their proofs can be found in ([AS], prop. 1.5).

**Proposition 3.1.** [AS] Let $(M, \mathcal{F})$ be a foliation, and fix a point $x \in M$. Let $X_1, ..., X_k \in \mathcal{F}$ be vector fields such that their images in $\mathcal{F}_x$ form a basis of $\mathcal{F}_x$. Then these vector fields span $\mathcal{F}$ in a neighbourhood $U$ of $x$.

*Proof.* By definition, we can take a generating set $Y_1, ..., Y_N$ in $\mathcal{F}$, which generate $\mathcal{F}$ in $U_1$. By assumption, one can write the image of each $Y_l$ in $\mathcal{F}_x$ as the linear combination $\sum_{i=1}^k \nu_{l,i} X_i$. In particular, $Y_l - \sum_{i=1}^k \nu_{l,i} X_i$ is an element of $I_x\mathcal{F}$. Using the fact that the $Y_j$ generate, we can find functions $\alpha_{j,l}$ in $I_x$ such that $Y_i - \sum_{i=1}^k \nu_{l,i} X_i = \sum_{j=1}^N \alpha_{i,j} Y_j$ for some neighbourhood of $x$. Thus, we have the equality $\sum_{i=1}^k \nu_{l,i} X_i = \sum_{j=1}^n \beta_{i,j} Y_j$ for $\beta_{i,j} = \delta_{i,j} - \alpha_{i,j}$. In matrix form, we find

$$B_y Y(y) = A_y X(y).$$

Notice that the off-diagonal arguments of $B$ vanish in $x$, whilst the diagonal elements are 1. Hence, $B_x = Id$, and since the determinant is continuous we can find a neighbourhood $U$ where $B$ is invertible. However, this implies that each $Y_j$ can be written as a linear combination of the $X_i$, from which the result follows. □

**Remark.** Notice that in the proof, we have proved that the dimension of $\mathcal{F}_x$ gives us the minimal amount of generators needed to generate $\mathcal{F}$ near $x$.

**Proposition 3.2.** [AS] The dimension of $\mathcal{F}_x$ is upper semi-continuous, and the dimension of $F_x$ is lower semi-continuous. Recall that a function $f$ is upper semi-continuous at $x_0$ if for every $y > f(x_0)$, we can find a neighbourhood $U$ of $x_0$ such that $f(x) < y$ for all $x \in U$.

*Proof.* By previous proposition, we have a neighbourhood $U$ of $x$ which is generated by a set of $\dim \mathcal{F}_x$ generators. This gives an upper bound of the amount of generators needed, and hence an upper bound for $\dim \mathcal{F}_y$ for $y \in U$. This is the desired neighbourhood in the definition of semi upper-continuity. For the second part of the proposition, we have that (using the notation of the proof of previous proposition) $F_y$ is spanned by the vectors $Y_1(y), ..., Y_N(y)$. Thus, the dimension of $F_y$ is the rank of the matrix $T(y)$ spanned by the $Y_j(y)$. Recall that the rank of a continuous map $y \mapsto T_y$ is lower semi-continuous, from which the result follows. □

This gives us some topological insight for leaves. We first need some terminology.

**Definition 3.4.** [AZ3] Let $(M, \mathcal{F})$ be a foliated manifold. A leaf $F$ is regular if there is a neighbourhood $U$ around $F$ for which every leaf $F'$ intersecting $U$, $F$ and $F'$ have the same dimension. A leaf that is not regular is called singular.

Using this terminology, we find that the set of regular leaves is dense and open.

**Example 3.1.** To avoid confusion, consider the foliation $f\frac{\partial}{\partial x}$ on $\mathbb{R}$, where $f$ is a function vanishing in $x \leq 0$, but $f(x) > 0$ for every $x > 0$. Then the origin is the only singular leaf of the foliation. Hence, being a singular leaf is relative only to nearby leaves, not to the global foliation.

We now look at some examples of singular foliations.

**Example 3.2.** Given any globally defined vector field $X$, the submodule it spans is a singular foliation. The associated partition of the manifold comes from the integral curves of $X$.

**Example 3.3.** Consider the action of $S^1$ on $\mathbb{R}^2$. This Lie group action is infinitesimally generated by the rotational vector field $x\partial_y - y\partial_x$. The induced partition of $\mathbb{R}^2$ is shown in figure 3.1. The submodule generated by this vector field is easily seen to be a singular foliation. In general, any Lie group action gives rise to a singular foliation. The associated partition comes from the orbits of the Lie group action.

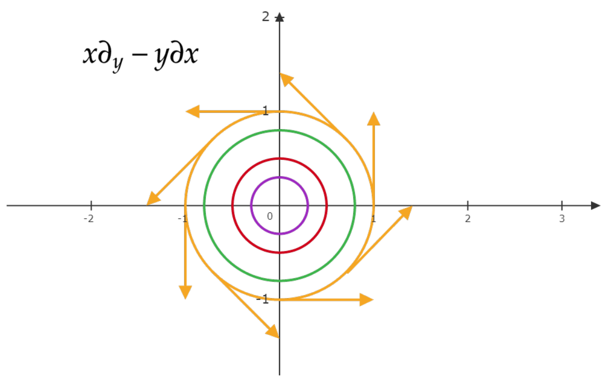



Figure 3.1: The partition due to the action of $S^1$ on $\mathbb{R}^2$

**Example 3.4.** ([AS], ex. 1.3.1) Let $\# : A \to TM$ be a Lie algebroid. Recall that when the anchor map was taken to be injective, one had that $\#(C^\infty(A, TM))$ was a regular foliation. We now claim that in the general case, $\mathcal{F}_A := \#(C^\infty(A, TM))$ is a (possibly singular) foliation. For this, notice that the sections of a vector bundle are always locally finitely generated. Since the anchor map on the space of sections is a Lie algebra homomorphism, it preserves the bracket. Hence, the image is indeed a locally finitely generated submodule, stable under the Lie bracket. Hence, to any Lie algebroid there is an associated singular foliation. In particular, any Lie groupoid gives rise to a singular foliation, since every Lie groupoid has a corresponding Lie algebroid.

**Example 3.5.** Any regular foliation is a singular foliation. Indeed, it is easy to see that the sections of the defining distribution satisfy all the necessary properties.

### 3.1.2 Basic results

Let $\mathcal{F}$ be a foliation on $M$, and let $g : M \to N$ be a diffeomorphism. Recall that this induces an isomorphism of modules: $g_* : C_c^\infty(M, TM) \to C_c^\infty(N, TN)$, associating to each vector field on $M$ its unique $g-$related vector field on $N$.

**Proposition 3.3.** [AS] Consider the setting above, then the image of $\mathcal{F}$ under $g_*$ is a foliation of $N$. Furthermore, we have

$$g_*(\mathcal{F})_{g(x)} \cong \mathcal{F}_x$$

for each $x \in M$.

*Proof.* Evidently, the module $g_*(\mathcal{F})$ is locally finitely generated by the images of local generators of $\mathcal{F}$. Using the basic result about Lie brackets of $g-$related vector fields, involutivity easily follows. Hence, $g_*(\mathcal{F})$ is indeed a foliation on $N$. By definition of $g$, we easily find that $(g_*(F))_{g(x)} = dg_x(F_x)$. The result then follows from the fact that $g_*(I_x\mathcal{F}) = I_{g(x)}g_*(\mathcal{F})$. □

Hence, foliations are well-behaved under diffeomorphisms. This allows us to compare foliations. There are two important groups of diffeomorphisms.

**Remark.** Recall that for a vector field $X$, we denoted by $\exp X$ its time-1 flow. In the setting of singular foliations, all vector fields considered have compact support. Hence, their flow is globally defined.

**Definition 3.5.** [AS]

1. The group $Aut(M, \mathcal{F})$ of diffeomorphisms of $M$ such that $g_*(\mathcal{F}) = \mathcal{F}$ for all $g \in Aut(M, \mathcal{F})$.
2. The group $\exp \mathcal{F}$ generated by $\exp X$ for $X \in \mathcal{F}$.

An important result, whose statement and proof can be found in [AS], is that $\exp \mathcal{F}$ is actually a (normal) subgroup of the automorphism group $Aut(M, \mathcal{F})$. An alternative proof can be found in [YG].

**Proposition 3.4.** [AS] The group $\exp \mathcal{F}$ is a normal subgroup of $Aut(M, \mathcal{F})$.

Recall that in the case of regular foliations, flowing along the generators of the distribution traced out the leaves of the foliation. This idea can be extended.

**Definition 3.6.** [AS] The leaves of a singular foliation $\mathcal{F}$ are the orbits of the natural action of $\exp \mathcal{F}$ on $M$.

Following proposition tells us that it makes sense to talk about the dimension of a leaf, and furthermore that the dimension of the fiber is also constant along the leaf.

**Proposition 3.5.** [AS] Let $x, y$ be two points on the same leaf $L$ of a foliated manifold $(M, \mathcal{F})$. Then $\dim F_x = \dim F_y$, and $\dim \mathcal{F}_x = \dim \mathcal{F}_y$. In particular, $\dim F_x$ and $\dim \mathcal{F}_x$ are constant on the leaves.

*Proof.* By definition, we can find a diffeomorphism $g$in $\exp\mathcal{F}$ such that $g(x) = y$. By the proof in proposition 3.3, $F_y = dg(F_x)$. Since $g$ is a diffeomorphism, $dg$ is an isomorphism. Again, by proposition 3.3, $\mathcal{F}_x$ and $\mathcal{F}_y$ are isomorphic. □

The following result can be found in [AZ3].

**Proposition 3.6.** [AZ3] Let $L$ be a leaf of $\mathcal{F}$. Then the infinitesimal isotropy $\mathfrak{g}$ is constant (up to isomorphism) along leaves. Furthermore, the Lie algebra is trivial if and only if $L$ is regular.

*Proof.* We first show the first statement. Let $x, y$ be two points on $L$. Let $g$ be diffeomorphism in $\exp\mathcal{F}$ satisfying $g(x) = y$. It suffices to show that $g_*(\mathcal{F}(x)) = \mathcal{F}(y)$, which follows immediately from the fact that $g$ preserves the foliation.

Let us now show the second statement. Assume that this Lie algebra is trivial at some point $x \in L$ (and hence on the whole of $L$). Let $U$ be a neighbourhood around $x$, in which $\mathcal{F}$ is generated by $X_1, ..., X_k$. Here, we take $k = \dim\mathcal{F}_x$ so this generating set is minimal by proposition 3.1. Shrinking $U$ if necessary, lower semi-continuity of $F_x$ implies that $\dim F_x \leq \dim F_y$ for all $y \in U$. Suppose for the sake of contradiction that this inequality is sharp. Since the image of the vector fields $X_1, ..., X_k$ at each $y \in U$ is exactly $F_y$, the inequality tells us that $\dim F_x < k$. Thus, there is a (non-trivial) linear combination $\sum_{i=1}^k a_i X_i(x) = 0$. We can extend this to a vector field $Y = \sum_{i=1}^k \alpha_i X_i$, where $\alpha_i(x) = a_i$. This vector field vanishes in $x$ and lies in $\mathcal{F}$, which implies $Y \in \mathcal{F}(x)$. By assumption, it thus holds that $Y \in I_x\mathcal{F}$. Thus under the quotient map $\mathcal{F}(x) \to \mathcal{F}_x$ , which is $\sum a_i[X_i]$, $Y$ gets mapped to 0 . Since the $[X_i]$ form a basis of $\mathcal{F}_x$ (due to the minimality condition), it must hold that all $a_i = 0$, which yields the desired contradiction. For the other direction, suppose $L$ is a regular leaf. By definition, we can find an open neighbourhood around $x \in L$ such that each leaf intersecting $L$ has the same dimension. It is easy to see that a minimal generating set in this neighbourhood will be linearly independent at $x$, hence $\mathcal{F}$ is generated by independent nowhere vanishing vector fields near $x$. This readily implies that $\mathcal{F}(x) = I_x\mathcal{F}$. □

## 3.2 Transversal maps

Recall that in the case of regular foliations, we could pull back foliations along transversal maps. This can also be done in the case of singular foliations. For this, we first recall the definition of the pull-back of a module. Most of the information can be found in [Gd] and [AS].

**Definition 3.7.** [Wk] Let $\varphi : N \to M$ be a smooth map. Let $E \xrightarrow{\pi} M$ be a vector bundle on $M$. Then the pull-back bundle $\varphi^*(E)$ is the pull-back of the square

$$\begin{array}{ccc} \varphi^*(E) & \dashrightarrow & E \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ N & \xrightarrow{\varphi} & M \end{array}$$

In other words, the pull-back bundle is defined by

$$\varphi^*(E) = \{(n, e) \in N \times E | \varphi(n) = \pi(e)\} \subset N \times E.$$

It is a vector bundle over $N$. Furthermore, if $\mathcal{S}$ is a submodule of the module of sections $C^\infty(M, TM)$, the pull-back module $\varphi^*(\mathcal{S})$ is the submodule of the module of sections $C^\infty(N, \varphi^*(E))$ generated by $f \cdot \varphi^*(\xi)$ with $f \in C^\infty(N)$ and $\xi \in \mathcal{S}$.

**Definition 3.8.** [AS] Let $M, N$ be manifolds and $f : N \to M$ a smooth map. Suppose $\mathcal{F}$ is a foliation on $M$.

1. The pull-back of $\mathcal{F}$, denoted $f^{-1}(\mathcal{F})$, is the submodule

$$f^{-1}(\mathcal{F}) = \{X \in C_c^\infty(N, TN) | df(X) \in f^*(\mathcal{F})\}$$

   of $C_c^\infty(N, TN)$.

2. We say that $f$ is transverse to $\mathcal{F}$ if

$$f^*(\mathcal{F}) \oplus C_c^\infty(N, TN) \to C^\infty(N, f^*(TM)) : (\xi, \nu) \mapsto \xi + df(\nu)$$

   is onto.

This definition has a more familiar pointwise counterpart. Following proposition relates the notion of transverse maps to the previously defined notion for regular foliations (see definition 1.5).

**Proposition 3.7.** [Gd] Let $\phi : M \to N$ be a map to a foliated manifold $(N, \mathcal{F})$. Then this map is transverse to $\mathcal{F}$ if and only if $d\phi_x(T_xN) + F_{\phi(x)} = T_{\phi(x)}M$, for all $x \in N$.

*Proof.* First, assume that $\phi \pitchfork \mathcal{F}$. Fix some $x \in N$. Let $y \in T_xM$, then we need to show that there exists vector fields $X_\phi \in C_c^\infty(N, TN)$ and $X_\mathcal{F} \in \mathcal{F}$ such that $d\phi(X_\phi)(x) + X_\mathcal{F}(x) = y$. Extend $y$ to some vector field $Y$ on $M$. Then, we can view $Y \circ \phi$ as a map from $N$ to $TM$, which we view as a section $X_Y$ of the vector bundle $C_c^\infty(N, \phi^*(TM))$. By transversality, we know that there exists a pair $X_\mathcal{F}$ and $X_\phi$ such that $d\phi(X_\phi) + X_\mathcal{F} = X_Y$. Thus, evaluating in $x$ gives us the desired result. For the other direction, assume that the map is not transversal. Let $\xi$ be a section in $C_c^\infty(N, \phi^*(TM))$ that does not lie in the image of the map in the definition of transversality. For any point $x \in M$, choose a basis $X_1, ..., X_n$ of $\mathcal{F}_x$, which we extend to local generators of $\mathcal{F}$ in some open $U$. Since $\phi$ is continuous, $\phi^{-1}(U)$ is open. By shrinking if necessary, we can obtain a local frame of $TN$ in $\phi^{-1}(U)$. The image of this frame under $d\phi$ generate the image of $d\phi$. Since $\xi$ is compactly supported, we can cover the support by finitely many opens $U_j$ such that $\mathcal{F}|_{U_j}$ is finitely generated for each $U_j$. For the sake of contradiction, suppose that we can decompose $T_{\phi(x)}M$ for all $x$ as in the proposition. Then, in any of the open subcovers, the local generators of $\mathcal{F}$ together with the image of the local frames of $TN$ span the whole of $TM|_{U_j}$ for each $j$. However, this would imply that in each $U_j$, we can write $\xi|_{U_j}$ in terms of these. Using a partition of unity subordinate to our cover, this would contradict the assumption. ☐

Following results regarding pullback foliations are the contents of proposition 1.10 and proposition 1.11 in [AS].

**Proposition 3.8.** [AS] Let $M, N$ be two manifolds, $f : M \to N$ smooth and $\mathcal{F}$ a foliation on $M$.

1. The $C^\infty(N)-$module $f^{-1}(\mathcal{F})$ is stable under Lie brackets
2. If $f$ is transverse to $\mathcal{F}$, $f^{-1}(\mathcal{F})$ is locally finitely generated.

*Proof.* The first part of the proposition is a straightforward calculation. For the second part, by the local nature of the statement we are allowed to restrict ourselves to small opens in $M$ and $N$. Since $\mathcal{F}$ is locally finitely generated, we can choose a small open neighbourhood $W$ around every $f(x)$ such that $\mathcal{F}$ is finitely generated when restricted to this open. Choosing $V$ and $U$ opens in $N, M$ respectively (where $U$ is chosen as a subset of $W$) small enough, we may assume that the vector bundles $TN \to N$ and $TM \to M$ are trivial bundles. Since the pullback of a trivial vector bundle is again trivial, the vector bundle $f^*(TM) \to N$ is again trivial. Notice that the submodule $f^{-1}(\mathcal{F})$ can be described by the fibered product

$$C_c^\infty(N, TN) \times_{C_c^\infty(N, f^*(TM))} f^*(\mathcal{F}).$$

Using Serre-Swan ([Sw]), one has that trivial vector bundles correspond to projective modules which implies in particular that $C_c^\infty(N, f^*(TM))$ is a projective module. Therefore, in the short exact sequence

$$0 \longrightarrow K \longrightarrow C_c^\infty(N, TN) \oplus f^*(\mathcal{F}) \longrightarrow C_c^\infty(N, f^*(TM)) \longrightarrow 0$$

the surjective function splits. Thus, we can find a section $s : C_c^\infty(N, f^*(TM)) \to C_c^\infty(N, TN) \oplus f^*(\mathcal{F})$. Notice that we can identify $K$ with $f^{-1}(\mathcal{F})$. However, we can also identify $K$ with the quotient $(f^*(\mathcal{F}) \oplus C_c^\infty(N, TN))/s(C_c^\infty(N, f^*(TM)))$: Any $(\alpha, \beta)$ in the quotient gets mapped to (the class of) $(\alpha, \beta)$, and in the opposite direction: suppose $\xi = \alpha + df(\beta)$, then $(\alpha, \beta) - s(\xi)$ lies in $K$. Thus,

$$f^{-1}(\mathcal{F}) \cong \frac{f^*(\mathcal{F}) \oplus C_c^\infty(N, TN)}{s(C_c^\infty(N, f^*(TM)))},$$

which is a quotient of a finitely generated module and hence finitely generated. □

**Proposition 3.9.** [AS] Let $M, N$ be two manifolds, and $\mathcal{F}$ a foliation on $M$. Suppose that $\phi : N \to M$ is a smooth map, transverse to $\mathcal{F}$. Denote by $\mathcal{F}_N$ the pull-back foliation on $N$.

1. For all $x \in N$, we have $(F_N)_x = \left\{\xi \in T_xN | d\phi_x(\xi) \in F_{\phi(x)}\right\}$.
2. Let $P$ be a manifold, and $\psi : P \to N$ a smooth map. The map $\psi$ is transverse to $\mathcal{F}_N$ if and only if $\phi \circ \psi$ is transverse to $\mathcal{F}$, and furthermore we have $(\phi \circ \psi)^{-1}(\mathcal{F}) = \psi^{-1}(\mathcal{F}_N)$.

*Proof.* We shall prove only (1). Let $\nu \in (F_N)_x$. Let $X \in \mathcal{F}_N$ such that $X(x) = \nu$. Then $(d\phi)_x(X(x)) = (d\phi)_x(\nu) \in F_{\phi(x)}$.

Suppose that $\xi \in T_xN$ such that $d\phi_x(\xi) \in F_{\phi(x)}$. Let $X \in C_c^\infty(N, TN)$ such that $X(x) = \xi$. Consider $d\phi(X)$,which is an element of $C_c^\infty(N, \phi^*(TM))$. By transversality, we can find $Z \in C_c^\infty(N, TN)$ and $Y \in \phi^*(\mathcal{F})$ such that $d\phi(Z) + Y = d\phi(X)$. Since $d\phi(\xi) \in F_{\phi(x)}$, we may assume that $Y(x) = d\phi_x(\xi)$, and thus that $Z(x) = 0$. By linearity of the differential, one finds that $d\phi(X - Z) = Y$. In other words: $X - Z \in \phi^{-1}(\mathcal{F})$. But, by construction, $(X - Z)(x) = \xi$. Therefore, $\xi \in (F_N)_x$. This proves (1). □

## 3.3 The local picture

In the case of Poisson structures, locally every Poisson structure splits in two "parts": its singular part and its regular part (i.e its symplectic part). This holds also for singular foliations. In this section, we shall prove a weaker result, whose statement and proof can be found in ([AS], prop. 3.10).

**Proposition 3.10.** [AS] Let $(M, \mathcal{F})$ be a foliated manifold, and let $x \in M$. Denote $k = \dim F_x$, and $q = \dim T_xM - k$.

1. There exists an open neighbourhood $W$ of $x$ in $M$, a foliated manifold $(V, \mathcal{F}_V)$ of dimension $q$ and a submersion $\phi : W \to V$ with connected fibers such that $\mathcal{F}_W = \phi^{-1}(F_V)$, where $\mathcal{F}_W$ is the restriction of $\mathcal{F}$ to $W$.

2. The tangent space of the leaf of $(V, \mathcal{F}_V)$ at the point $\phi(x)$ is 0, we have $\ker(d\phi)_x = F_x$ and each fiber of $\phi$ is contained in a leaf of $(M, \mathcal{F})$.

*Proof.* We first prove (1) via induction on $k$. For the base case, if $k = 0$, it suffices to take $\phi$ as the identity on $M$. Suppose that the induction hypothesis holds for all $j$ up to $k-1$. Since $k \neq 0$, there is a non-zero vector $\xi \in F_x$, which we may extend to a vector field $X \in \mathcal{F}$. Denote by $\phi_t$ the one-parameter group of diffeomorphisms $\exp(tX)$. Recall that this is a family of automorphisms of the foliation. Let $V_0$ be a locally closed submanifold of $M$ containing $x$ such that $T_xV_0 \oplus \mathbb{R}X(x) = T_xM$. The flow map $F : \mathbb{R} \times V_0 \to M$ is smooth and one has $F(0, x) = x$ (since $\exp(0) = Id$) and $(dF)_0(s, \xi) = sX(x) + \xi$ for every $s \in \mathbb{R}$ and $\xi \in T_xV_0$. Since $X(x)$ and $T_xV_0$ spanned the whole of $T_xM$, the differential $(dF)_0$ is bijective. Hence, from the local inverse theorem, we can find an open neighbourhood around $(0, x)$ for which it is a diffeomorphism onto its image, i.e we find a diffeomorphism $\psi : I \times V \to W$. Denote by $\phi : W \to V$ the composition $pr_V \circ \psi^{-1} : W \to V$. This map takes a point $w$ in $W$ and maps it to the (unique) point in $V$ whose flow line contains $w$. Denote by $\mathcal{F}_W$ the restriction of $\mathcal{F}$ to the open subset $W$. Since $V$ is transverse to $\mathcal{F}$, the submodule $\mathcal{F}_V = i^{-1}(\mathcal{F})$ is a foliation on $V$. (Here, $i$ is the inclusion of $V$ into $M$). Since $W \cong I \times V$, we can decompose any vector field $Y$ on $W$ (uniquely) as $Y = fX + Z$. Here, $Z$ is "parallel" to $V$, in the sense that (identifying $w$ with $(t, v)$) one has $Z(t, v) = \phi_t(Z'_t(v))$ where $Z'_t$ is tangent to $V$. Notice now that $Y \in \mathcal{F}_W$ if and only if $Y - fX \in \mathcal{F}_W$ (since $fX \in \mathcal{F}$, its restriction lies in $\mathcal{F}_W$). Hence, $Y$ lies in $\mathcal{F}_W$ if and only if $Z$ lies in $\mathcal{F}_W$. Since the flow of $X$ preserves the foliation structure, this is equivalent to $Z'_t$ lying in $\mathcal{F}_V$ for all $t \in I$. Hence, $\mathcal{F}_W = \phi^{-1}(\mathcal{F}_V)$. Evidently, $(F_V)_x$ has dimension $k-1$. Therefore, we may apply the induction hypothesis to $V$. We now prove (2).

The first statement easily follows from the fact that the codimension of $F_x$ is exactly the dimension of $V$. The second statement is then a straightforward application of proposition 3.9. For the final statement, each vector field on $W$ tangent to the fibers of $\phi$ gets mapped to 0 by $d\phi$. Since the zero vector field lies in $\mathcal{F}_V$, this vector field lies in $\phi^{-1}(\mathcal{F}_V) = \mathcal{F}_W$. □

**Remark.** This theorem shows that singular foliations have the following local picture. By the submersion theorem, we can consider $(W, \mathcal{F}_W)$ and $(V, \mathcal{F}_V)$ as open subsets of $\mathbb{R}^n$ and $\mathbb{R}^q$ and $\pi$ as the projection map of $\mathbb{R}^n$ onto $\mathbb{R}^q$. Then $\phi^{-1}(\mathcal{F}_V) = \mathcal{F}_W$, i.e the foliation is generated by the first $k$ coordinate vector fields (which are tangent to the fibers) and $\mathcal{F}_S$, which is a foliation of $\mathbb{R}^q$ vanishing at the origin.

## 3.4 Leafwise smooth struture

In this section, we study the smooth structures of leaves, as given by Androulidakis and Skandalis in ([AS], section 1.3.2). This is done by considering a particular smooth structure on $M$, which is inherited from the foliation. In this smooth structure, every leaf is an open smooth submanifold.

We will need the following concept.

**Definition 3.9.** Let $f : N \to M$ be a smooth map, and $\mathcal{F}$ a foliation on $M$. We say that $f$ is leafwise if $f^{-1}(\mathcal{F}) = C_c^\infty(N, TN)$.

**Remark.** A reformulation of this definition is that $f$ is leafwise if

$$df(C_c^\infty(N, TN)) \subset f^*(\mathcal{F}),$$

Notice that when $f : N \to M$ is leafwise, and $N$ is connected, then $f(N)$ is contained in a leaf of $\mathcal{F}$.

In ([AS], prop. 1.14), the authors show the following.

**Proposition 3.11.** There is a new smooth structure on $M$ called the leafwise structure such that a map $f : N \to M$ is smooth if and only if it is smooth and leafwise. The leaves are the connected components of this structure.

In this smooth structure, the tangent space $T_xM$ for any $x \in M$ is exactly $F_x$, where $F$ is the leaf through $x$.

## 3.5 Transitive Lie algebroid on leaves

In this section, we briefly mention the existence of a Lie algebroid $A_L$, together with a nice result due to Debord [CD]. The Lie algebroid is obtained by "gathering all fibers $\mathcal{F}_x$ along the leaf".

**Definition 3.10.** Let $(M, \mathcal{F})$ be a foliated manifold. Fix a leaf $L$, then the Lie algebroid $A_L$ associated to $L$ is given by

$$A_L = \cup_{x \in L} \mathcal{F}_x.$$

Its space of sections is given by $\Gamma_c(A_L) \cong \frac{\mathcal{F}}{I_L \mathcal{F}}$, where $I_L$ is the set of compactly supported functions vanishing on $L$. In the next chapter, we will define the holonomy groupoid $H$ of a singular foliation $\mathcal{F}$. This groupoid will be related to $A_L$ via the following result. Following proposition tells us that the holonomy groupoid is *longitudinally smooth*, in the sense that the restriction of $H$ to a leaf is a Lie groupoid. Its statement and proof can be found in ([CD2], prop. 2.2).

**Proposition 3.12.** [CD2] Let $(M, \mathcal{F})$ be a foliated manifold, and fix a leaf $L$. Consider the groupoid $H_L$, which is the restriction of the holonomy groupoid to $L$. Then $H_L$ is a Lie groupoid, and will integrate the Lie algebroid $A_L$.

Of course, we don't yet know what the holonomy groupoid of a singular foliation looks like. This is the content of the next chapter.

# Chapter 4

# The holonomy groupoid of singular foliations

Recall that any Lie groupoid $G \rightrightarrows M$ defines a foliation on $M$. This foliation is the image of the anchor of the associated Lie algebroid $A$ of $G$. Recall that for regular foliations, we have seen the opposite direction too: any regular foliation has an associated Lie groupoid, called the holonomy groupoid. In this chapter, we will construct a groupoid $G$ which "integrates" more general singular foliations. This construction was first found by Androulidakis and Skandalis in their paper *The holonomy groupoid of singular foliations*, see [AS]. Hence, many of the following results can be found in this paper, but the order in which they are stated and the interpretations of the results are different. Other helpful sources are [Gd] and [W].

## 4.1 Bisubmersions and bisections

### 4.1.1 Bisections

Due to the presence of singularities, the path approach we used in the case of regular foliation no longer works. To see why, we consider an example.

**Example 4.1.** Consider the foliation on $\mathbb{R}^2$ induced by the vector field $x\partial_x$. A transversal section at the origin is an open neighbourhood of the origin, since a transversal section $S$ through 0 must satisfy

$$T_0S \oplus F_0 = T_0S = T_0\mathbb{R} \cong \mathbb{R}.$$

Let us naively try to extend the path approach for this (singular) case. For this, we need to choose paths in nearby leaves. Notice that there are many possible choices: one choice is the constant path at every point. The induced germ of the diffeomorphism associated to this choice, i.e diffeomorphism that maps the begin points in leaves to the endpoints of the paths chosen, is the identity map on $S$. Another possible choice of paths is the path $(z,t) \mapsto \phi^t_{x\partial x}(z)$, where $\phi^t_{x\partial x}$ is the time-$t$ flow of the vector field $x\partial_x$. The germ arising from this choice does not correspond to the germ of the identity diffeomorphism, as for example their derivatives at the origin do not coincide. In particular, the ambiguity of the choice of nearby paths is too large, in the sense that it does not induce a unique germ of diffeomorphism on $S$.

Due to this ambiguity, we will need to find another way to define the notion of holonomy. We will give a construction that is (at first sight) completely different than the construction given earlier. Before even beginning with the construction, it is crucial that one thinks about what the holonomy groupoid *should be*. For this, let us assume that our foliation $\mathcal{F}$ is defined by a given Lie groupoid $G \rightrightarrows M$. Interested in the information that $G$ encodes, we define the following.

**Definition 4.1.** Let $G \rightrightarrows M$ be a Lie groupoid. A bisection is a locally closed submanifold $V$ of $G$ such that the restrictions $s|_V$ and $t|_V$ are diffeomorphisms from $V$ onto open subsets of $M$. The local diffeomorphism associated to a bisection $V$ is the diffeomorphism $\phi_V = t \circ s^{-1}$.

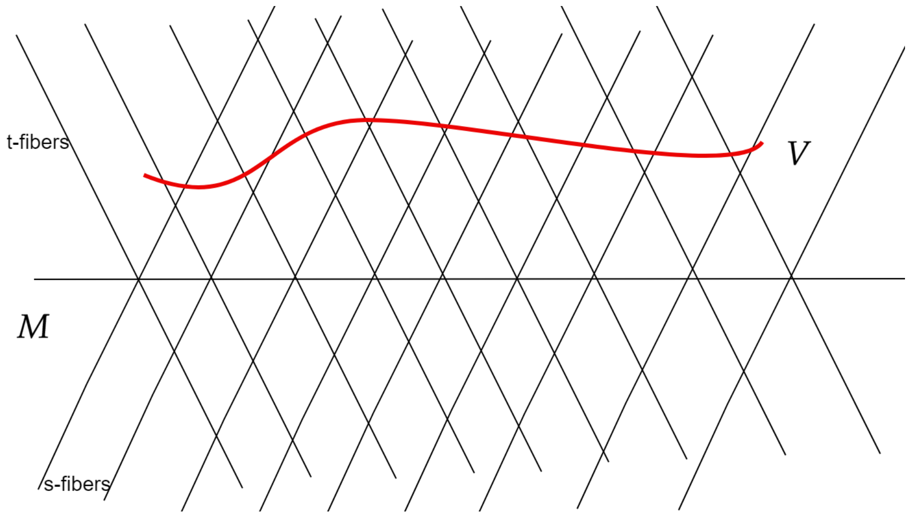



Figure 4.1: A bisection

In figure 4.1, one sees a sketch of what such a bisection looks like. The associated diffeomorphism is then obtained by mapping a point $s(v)$ to $t(v)$, where $v \in V$. Here, we used the graphical representation of a groupoid seen earlier. The geometric information encoded by such bisections is the following: for a groupoid $G \rightrightarrows M$ defining a foliation $\mathcal{F}$, the bisections are local diffeomorphisms contained in $\mathrm{Aut}(M, \mathcal{F})$. Bisections of such a groupoid tell us how we can move along the leaf. In the regular case, the local bisections of the holonomy groupoid encode $\exp \mathcal{F}$. This gives us a direction we want to work towards: we want a groupoid that encodes $\exp \mathcal{F}$. These bisections are local objects, and we will in fact tackle the problem as if it were local. In what follows, we will define *bisubmersions*. These objects will resemble groupoids, and should be seen as 'pieces of the holonomy groupoid'.

### 4.1.2 Bisubmersions

Before stating the definition, we briefly motivate where these objects originate from. Given a manifold $M$, there is a trivial foliation on $M$: the only leaf is $M$ itself. The foliation $\mathcal{F}$ is then (locally) generated by a choice of frame $\partial x_1, ..., \partial x_n$ associated to a choice of local coordinates. To describe the leaf (and hence the foliation), we use the flows of these vector fields. This can be encoded in a neighbourhood $W \subset U \times \mathbb{R}^n$ (where $U$ is a coordinate neighbourhood) as follows:

$$s, t : W \to M : s = pr_U, \quad t : (p, y_1, ..., y_n) \mapsto \exp(\sum y_i \partial x_i)(p).$$

Recall that we defined $\exp(X)$ to be the time-1 flow of $X$. This object is not a groupoid: there is no partially defined multiplication present. However, we can still talk about bisections of $W$: these are the local diffeomorphisms corresponding to the flows $\exp \mathfrak{X}(M)$. Thus, this object can be used to record locally defined diffeomorphisms respecting the (albeit trivial) foliation.

Let us now give the concrete definition.

**Definition 4.2.** [AS] Let $(M, \mathcal{F})$ be a foliated manifold. A bisubmersion of $(M, \mathcal{F})$ is a triple $(U, t, s)$ with $U$ a smooth manifold and two submersions $t, s : U \to M$ satisfying

1. $s^{-1}(\mathcal{F}) = t^{-1}(\mathcal{F})$.
2. $s^{-1}(\mathcal{F}) = C_c^\infty(M, \ker(ds)) + C_c^\infty(M, \ker(dt))$.

**Remark.** Since $t, s$ are submersions, we know from proposition 3.8 that $s^{-1}(\mathcal{F})$ and $t^{-1}(\mathcal{F})$ are foliations on $U$. This definition forces the induced foliations to be the same. The second condition tells us explicitly what this foliation looks like. Notice that $C_c^\infty(M, \ker(ds))$ and $C_c^\infty(M, \ker(dt))$ are sections of a vector bundle, and hence are the sections of constant rank distributions. However, in general their sum is not and therefore the foliation $s^{-1}(\mathcal{F})$ on $U$ need not be regular.

To motivate this definition, we consider the following proposition ([AS], prop. 2.2).

**Proposition 4.1.** [AS] Let $G \rightrightarrows M$ be a Lie groupoid, and $\mathcal{F}$ be its associated foliation. Then $(G, t, s)$ is a bisubmersion of $(M, \mathcal{F})$.

*Proof.* See ([AS], pp. 11). □

Thus, a bisubmersion can be seen as a generalisation of a Lie groupoid. As we mentioned earlier, we can talk about the bisections of a bisubmersions. For the sake of completeness, we state the definition.

**Definition 4.3.** [AS] Let $(M, \mathcal{F})$ be a foliated manifold, and $(U, t, s)$ a bisubmersion of $(M, \mathcal{F})$. Let $x \in s(U)$. Then

1. A bisection at $x$ is a locally closed submanifold $V$ of $U$ such that the restrictions of both $s$ and $t$ to $V$ are diffeomorphisms from $V$ to open subsets of $M$.
2. The local diffeomorphism associated to $V$ is $\phi_V = t_V \circ s_V^{-1}$.
3. Let $u \in U$ and $\phi$ a local diffeomorphism of $M$. Then $\phi$ is said to be carried by $(U, t, s)$ at $u$ if there exists a bisection $V$ containing $u$ such that $\phi_V$ coincides with $\phi$ in a neighbourhood of $u$.

The bisections of a bisubmersion should be viewed as the bisections of our 'atlas' seen in the introduction. For this to make sense, we should of course expect the obtained diffeomorphisms to preserve the foliation. The next result tells us that this is the case.

**Proposition 4.2.** Let $(M, \mathcal{F})$ be a foliated manifold, and $(U, t, s)$ a bisubmersion of $(M, \mathcal{F})$. Let $V$ be a bisection of $(U, t, s)$. Then the associated local diffeomorphism $\phi_V$ preserves the foliation, i.e $(\phi_V)_*(\mathcal{F}) = \mathcal{F}$.

*Proof.* Equivalently, we have to show that $(\phi_V)^{-1}(\mathcal{F}) = \mathcal{F}$. Using the definition, one can write $(\phi_V)^{-1}(\mathcal{F}) = (t_V \circ s_V^{-1})^{-1}(\mathcal{F}) = s_V(t_V^{-1}(\mathcal{F}))$. However, from the definition of a bisubmersion, one has $t_V^{-1}(\mathcal{F}) = s_V^{-1}(\mathcal{F})$. Thus, one finds $(\phi_V)^{-1}(\mathcal{F}) = s_V(s_V^{-1}(\mathcal{F})) = \mathcal{F}$, proving the desired result. □

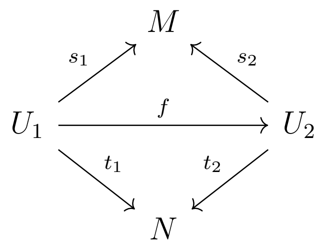
**Remark.** Thus, we find that the local diffeomorphisms recorded by bisubmersions indeed preserve the foliation structure.

To motivate the next result, recall that every point $(x, y_1, ..., y_n)$ could be interpreted as a local diffeomorphism by choosing a bisection: one extended this point to a locally closed submanifold containing it and looked at the induced diffeomorphism. Thus, we would want bisubmersions to have a lot of bisections. Luckily, this is the case.

**Proposition 4.3.** [AS] Let $(U, t, s)$ be a bisubmersion of a foliated manifold $(M, \mathcal{F})$. Let $u \in U$, then there exists a bisection $V$ containing $u$.

We can compare bisubmersions, as following definition shows.

**Definition 4.4.** [W] A <u>morphism</u> of bisubmersions $(U_1, t_1, s_1)$ and $(U_2, t_2, s_2)$ between $(M, \mathcal{F})$ and $(N, \mathcal{F}')$ is a smooth map $f : U_1 \to U_2$ such that the following diagram commutes:




In other words, we have $s_1(u) = s_2(f(u))$ and $t_1(u) = t_2(f(u))$ for all $u \in U$. There is also a notion of local morphism between bisubmersions, where we replace $f : U_1 \to U_2$ with a map $f : U' \to U_2$, where $U'$ is an open subset of $U$.

A bisubmersion is a local object. To show this, we will need the following lemma.

**Lemma 4.1.** Let $\{W_i\}$ be an open cover of $M$ and $\mathcal{F}_i$ a foliation on every $W_i$. Suppose that $\mathcal{F}_i = \mathcal{F}_j$ on $W_i \cap W_j$ for all $i, j$. Then there is a unique foliation $\mathcal{F}$ on $M$ such that $\mathcal{F}_i = \mathcal{F}|_{W_i}$.

*Proof.* Let $\{\phi_i\}_i$ be a partition of unity subordinate to the covering by $W_i$. This allows us to extend vector fields in every $\mathcal{F}_i$ to a global vector field on $M$ in the usual way. Consider the newly obtained submodule of sections of $TM$, generated by these vector fields. Let us show that this gives rise to a foliation. By construction, this is easily seen to be locally finitely generated. To show that it is involutive, choose $X_i \in \mathcal{F}_i$ and $X_j \in \mathcal{F}_j$. We need to show that $[\phi_i X_i, \phi_j X_j] \in \mathcal{F}$. For this, we have

$$[\phi_i X_i, \phi_j X_j] = \phi_i \phi_j [X_i, X_j] + \phi_i(X_i(\phi_j))X_j - \phi_j(X_j(\phi_i))X_i.$$

This identity follows from the basic properties of the Lie bracket. In this form, using the fact that the two foliations coincide on their intersection, one can see that $\mathcal{F}$ is closed under the Lie bracket. To show uniqueness, we need to show that two foliations agree globally if and only if they agree locally. One direction is trivial, so let us assume they agree locally. This means that for every point $x \in M$, we can find a neighbourhood $U$ such that $\mathcal{F}|_U = \mathcal{F}'|_U$. Let $X \in \mathcal{F}$, then $X$ is a compactly supported vector field. Cover the support of $X$ with the aforementioned opens and choose a finite subcovering. In each $U_j$, we can find a vector field $\xi_j$ in $\mathcal{F}'|_U$ such that $\xi_j = X|_{U_j}$. Gluing these together with partitions of unity, we find that $X \in \mathcal{F}'$. We can repeat the argument for the other inclusion, yielding $\mathcal{F} = \mathcal{F}'$. □

**Proposition 4.4.** Let $(M, \mathcal{F})$ be a foliated manifold. Suppose $(U, t, s)$ is a triple, with $U$ a manifold and $s, t$ submersions $U \to M$. Suppose $\{U_i\}$ is a covering of $U$. Then $(U, t, s)$ is a bisubmersion if and only each $(U_i, t_i, s_i)$ is, where $s_i$ is the restriction of $s$ to $U_i$.

*Proof.* Suppose $(U_i, t_i, s_i)$ is a bisubmersion for all $i$. Since $s, t$ are submersions, they are transversal to $\mathcal{F}$, which implies that $s_i^{-1}(\mathcal{F}) = t_i^{-1}(\mathcal{F})$ is a foliation of $U_i$. Since $s_i|_{U_j} = s_j|_{U_i}$ and the same holds for $t_i, t_j$, the foliations agree on intersections. This implies by lemma 4.1 that there is a unique foliation $\mathcal{F}_U$ on the whole of $U$ which restricts to $s_i(\mathcal{F})$ on each $U_i$. Since $s^{-1}(\mathcal{F})$ and $t^{-1}(\mathcal{F})$ both restrict to these sets, uniqueness in above lemma shows that $s^{-1}(\mathcal{F}) = t^{-1}(\mathcal{F})$. The other direction is obvious. □

Following proposition tells us that bisubmersions can be inverted and composed.

**Proposition 4.5.** [AS] If $(U, t, s)$ is a bisubmersion, then so is $(U, s, t)$ (where we swapped the roles of $s$ and $t$). If $(U, t_U, s_U)$ and $(V, t_V, s_V)$ are bisubmersions, then $(U \times_{s_U, t_V} V, t_U, s_V)$ is a bisubmersion.

**Remark.** Recall that the set $U \times_{s_U, t_V} V$ is defined by

$$U \times_{s_U, t_V} V = \{(u, v) \in U \times V | s_U(u) = t_V(v)\}.$$

**Notation 4.1.** We sometimes denote the composition of two bisubmersions $(U_1, t_1, s_1)$ and $(U_2, t_2, s_2)$ by $U_1 \circ U_2$. Similarly, in the setting of above proposition, we sometimes denote $(U, s, t)$ as $U^{-1}$.

At this point, we have found desirable objects: bisubmersions are able to record the desired information, namely automorphisms of foliations. However, there are *a lot* of possible bisubmersions. Recall that the holonomy groupoid in the regular case was minimal. Since we want our holonomy groupoid to be the singular sibling of the regular case, it is nice to require minimality too. Hence, we would like to find special bisubmersions that record in some sense the 'minimal' amount of information, whilst still giving enough information for our purposes. This is the topic of next section.

## 4.2 Path-holonomy bisubmersions

Let us recall the motivational example of bisubmersions: using a local frame, one 'traced out' the leaf by flowing along the vector fields. Intuitively, $W = M \times \mathbb{R}^n$ where $x$ was the starting point and $(y_1, ..., y_n)$ were 'times', indicating how long one had to flow in the direction of the associated vector field. In this section, we would like to extend this idea. As in the introduction, one can intuitively think about these bisubmersions as telling us how one can move a point along a leaf. Of course, this extension requires a bit of work. Whilst we could find a base of vector fields of $\mathcal{F}$ in a neighbourhood $U$ (which was just a frame in our trivial example), this need not be the case for singular foliations due to the existence of singularities. Nevertheless, from proposition 3.1 we can find local generators $X_1, ..., X_n$ of $\mathcal{F}$ in some neighbourhood $U$. Using these, we will extend the idea given above to the singular case. The next theorem tells us that this indeed gives rise to bisubmersions. We will call these bisubmersions *path-holonomy bisubmersions.* They "correspond" to small exponentiations of generating vector fields of the foliation.

**Theorem 4.1.** [AS] Let $(M, \mathcal{F})$ be a foliated manifold, and $x \in M$. Let $X_1, ..., X_n$ be vector fields whose images in $\mathcal{F}_x$ form a basis of $\mathcal{F}_x$. For $y \in \mathbb{R}^n$, we define $\phi_y = \exp(\sum y_i X_i) \in \exp \mathcal{F}$. Write $W_0 = \mathbb{R}^n \times M$ and define $s_0(y, x) = x$ and $t_0(y, x) = \exp_x(\sum y_i X_i)$. Then

1. There is a neighbourhood $W$ of $(0, x)$ in $W_0$ making $(W, t, s)$ a bisubmersion. Here, $t, s$ are the restrictions of $t_0, s_0$ to $W$.
2. Let $(V, t_V, s_V)$ be a bi-submersion and $v \in V$. Assume that $s(v) = x$, and that the identity of $M$ is carried by $(V, t_V, s_V)$ at $v$. Then there is a local morphism of bi-submersions $g : V' \to W$ (with $V'$ an open neighbourhood of $V$) that is a submersion and satisfies $g(v) = (0, x)$.

*Proof.* We first prove (1). For this, consider the vector field $Z$ on $W_0$ defined by $Z(y, x) = (0, \sum_i y_i X_i)$. Evidently, since $s_0$ is the second projection, we have $Z \in s_0^{-1}(\mathcal{F})$. Hence, the flow of this vector field $\phi = \exp Z$, is an automorphism of the foliation $s_0^{-1}(\mathcal{F})$ on $W_0$. Consider now the reflection $\alpha : (y, x) \mapsto (-y, x)$, and consider $\kappa = \alpha \circ \phi$. Remark that $s_0^{-1}(\mathcal{F})$ is invariant under $\alpha$ (and thus under $\kappa$). Since $\kappa^2 = id$ (it corresponds to first flowing in one direction, then flowing in the opposite direction) and $s_0 \circ \kappa = t_0$ (which is straightforward), we find

$$t_0^{-1}(\mathcal{F}) = \kappa^{-1} \circ s_0^{-1}(\mathcal{F}) = \kappa \circ s_0^{-1}(\mathcal{F}) = s_0^{-1}(\mathcal{F}).$$

From this, we see that $C_c^\infty(W_0, \ker ds_0) + C_c^\infty(W_0, \ker dt_0) \subset s_0^{-1}(\mathcal{F})$. Indeed, it is easy to see that $C_c^\infty(W_0, \ker ds_0) \subset s_0^{-1}(\mathcal{F})$ and analogously $C_c^\infty(W_0, \ker dt_0) \subset t_0^{-1}(\mathcal{F})$. Since we just showed that $s_0^{-1}(\mathcal{F}) = t_0^{-1}(\mathcal{F})$, we conclude. Let us now consider $dt_0(\ker ds_0) \subset t^*(\mathcal{F})$. Since $\mathcal{F}$ is spanned by the $X_i$ near $x$, we can find a neighbourhood $W$ of $(0, x)$ in $W_0$ and a smooth function $h = (h_{ij})$ with values in the $n \times n$-matrices such that

$$(dt_0)_{y,u}(z, 0) = \sum z_i h_{i,j}(y, u) X_j,$$

where $(y, u) \in W$ and $z \in \mathbb{R}^n$. This follows from the fact that $dt_0(\ker ds_0) \subset \mathcal{F}$. Furthermore, we have $h_{ij}(0, x) = \delta_{ij}$. By shrinking $W$, we may assume that $h$ is invertible. We denote the restriction of $s_0$ and $t_0$ to $W$ by $s_W$ and $t_W$ repsectively. Thus, we have just proven that $(dt_W)(C_c^\infty(W, \ker ds_W)) = t_W^*(\mathcal{F})$. Hence, $t_W^{-1}(\mathcal{F}) \subset C_c^\infty(M, \ker ds_W) + C_c^\infty(W, \ker dt_W)$. This proves (1). To prove (2), notice that the required submersion is local. Hence, we can assume $V$ to be a small open subset containing $V$, which we shrink so that $s_V(V) \subset s(W)$. By shrinking if necessary, the bundles $\ker dt_V$

and $\ker ds_V$ are both trivial. Since being a bisubmersion is a local condition, we still have a bisubmersion $(V, t_V, s_V)$. Evidently the map $dt : C_c^\infty(V, \ker ds_V) \to t_V^*(\mathcal{F})$ is surjective. Thus, we can find $Y_1, ..., Y_n \in C_c^\infty(V, \ker ds_V)$ such that $dt_V(Y_i) = X_i$. Since $dt_V$ is linear, the fact that $X_i(x)$ form a basis of $\mathcal{F}_x$, we have that the $Y_i(v)$ are also independent. Since linear independence is a local condition, we may shrink $V$ (so that it still contains $v$) and assume the $Y_i$ are independent on the whole of $V$. Choose $Z_{n+1}, ..., Z_k$ so that $(Y_1, ..., Y_n, Z_{n+1}, ..., Z_k)$ form a frame of $\ker ds_V$ (recall that these bundles were trivial!) One can replace $Z_i$ with $Y_i$ to obtain a set $(Y_1, ..., Y_k)$ such that $t_V^*(Y_i) = X_i$ for $i \leq n$ and $t_V^*(Y_i) = 0$ if $i > n$ (using the fact that the $Y_1, ..., Y_n$ span the image of $dt_V^*$). Note that we can do this in such a way that $(Y_1, ..., Y_k)$ still form a frame. For small $y \in \mathbb{R}^k$, we get a partially defined diffeomorphism $\exp(\sum y_i Y_i)$ of $V$. By assumption, there is a bisection $U_0 \subset V$ through $v$ where $s_V$ and $t_V$ coincide. We can thus view $U_0$ as a subset in $\mathbb{R}^n$, by identifying it with its image under $s_V$ or equivalently $t_V$. We can find an open neighbourhood $U$ of $v$ in $U_0$ and a small open ball $B \subset \mathbb{R}^k$ such that $h : (y, u) \mapsto \psi_y(u)$ is a diffeomorphism of $U \times B$ into an open neighbourhood $V'$ of $v$. Let $p : \mathbb{R}^k \to \mathbb{R}^n$ be the projection map. The map $p \circ h^{-1} : V' \to W$ is the desired morphism, and is a submersion. □

As a corollary of theorem 4.1, we find the following proposition.

**Proposition 4.6.** Let $(U, t_U, s_U)$ and $(V, t_V, s_V)$ be bisubmersions. Let $u \in U$ and $v \in V$ be such that $s_U(u) = s_V(v)$.

1. If the identity local diffeomorphism is carried by (cf. definition 4.3) $U$ at $u$ and by $V$ at $v$, there exists a local morphism of bisubmersions $f : U' \to V$ such that $f(u) = v$.

2. If there is a local diffeomorphism carried by $U$ at $u$ and $V$ at $v$, then there exists a local morphism $f : U' \to V$ such that $f(u) = v$.

3. If there is morphism of bisubmersions $g : V \to U$ such that $g(v) = u$, then there is a local morphism of bisubmersions $f : U' \to V$ such that $f(u) = v$.

*Proof.* We first prove (1). Using the notation of theorem 4.1, we have local morphisms of bisubmersions $g : U' \to W$, $h : V' \to W$ which are submersions such that $g(u) = h(v) = (0, x)$. Recall that for a submersion, every point in the domain lies in the image of a local section. Let $h_s$ be a local section such that $h_s(0, x) = v$ of the submersion $h$. The domain of $h_S$ is an open subset around $(0, x)$. We can shrink the domain of $g$ (so that it still contains $u$) so that the image of $g$ is contained in the domain of $h_s$. Then consider the map $f = h_s \circ g$. We claim that this is a morphism of bisubmersions. Let $u' \in U$. Then $t_V(f(u)) = t_V(h_s(g(u))) = t_V(h_s(0, x)) = t_V(v)$. The proof for $s_V$ is completely analogous. This proves (1). To show (2), suppose $\vartheta$ is a local diffeomorphism carried by $U$ at $u$ and by $V$ at $v$. We may shrink $U, V$ so that $t_U(U)$ and $t_V(V)$ both lie in the image of $\vartheta$. Then consider $(U, \vartheta^{-1} \circ t_U, s_U)$ and $(V, \vartheta^{-1} \circ t_V, s_V)$. These carry the identity local diffeomorphism, and hence we may apply (1). One can easily deduce that this implies the proposition. For (3), it suffices by (2) to show that there is a local diffeomorphism carried by both $U$ and $V$. Suppose $V_0$ is the bisection through $v$ carrying $\phi$. Then $g(V_0)$ is a bisection through $u$ carrying $t_V \circ g \circ g^{-1} \circ s_V^{-1} = \phi$. Hence they carry the same diffeomorphism, so we conclude. □

**Remark.** Suppose that we have a neighbourhood $W$ of a point $(0, x)$ as in the theorem 4.1. Consider a local submanifold $V$ of the form $\{0\} \times U$, where $U$ is an open neighbourhood of $x$. By taking $U$ small enough, we can ensure $V \subset W$ since $W$ is open. This is a bisection through $(0, x)$: the source map embeds $V$ into $M$ and since the time-zero flow is the identity, the same holds for the target map. In particular, the carried diffeomorphism obtained from this bisection is the identity. For this reason, path-holonomy bisubmersions are sometimes called bisubmersions near the identity.

More importantly, these bisubmersions carry the diffeomorphisms in $\exp \mathcal{F}$. As remarked earlier, this was *exactly* the set of diffeomorphisms we wanted to be able to record! Thus, these bisubmersions have very desirable properties.

However, we are not yet finished since we still need to glue together these bisubmersions into a groupoid structure. This is the topic of the next section.

# 4.3 The holonomy groupoid

## 4.3.1 Introduction

In this section, we will show how to construct a groupoid out of a certain family of bisubmersions, called an atlas. The ideas presented are found in ([AS], section 3), but we choose to give a softer introduction to the topic in this subsection first. It is important to recall everything we have already done, and the things we still need to do. We have already found a very promising type of bisubmersion, called path-holonomy bisubmersions. We have seen that they encode $\exp \mathcal{F}$, which is what we deemed necessary in the beginning of this chapter. We have already briefly discussed that the holonomy groupoid had to encode the information regarding the composition of the relevant bisubmersions (to get a more "global" point of view). Hence, we end up with the following definition.

**Definition 4.5.** Let $(M, \mathcal{F})$ be foliated manifold. Let $\{(U_i, t_i, s_i)\}_{i \in I}$ be some family of path-holonomy bisubmersions covering $M$, i.e $\cup_{i \in I} s_i(U_i) = M$. Let $\mathcal{U}$ be the family containing all finite compositions of the $U_i$ and their inverses. Then we define the holonomy groupoid $H(\mathcal{F})$ (or just $H$ if the underlying foliation is clear) by

$$H(\mathcal{F}) = \bigsqcup_{U \in \mathcal{U}} U / \sim$$

with a suitable equivalence relation we will define later. We denote by $Q = (q_i)_{i \in I}$ the projection of the disjoint union onto $H$.

**Remark.** In this definition, we implicitly used the fact that any point of $M$ lies in a bisubmersion which is essentially the content of the first statement in theorem 4.1. One way to rephrase the definition, which might give a bit more geometric insight, is as follows. One covers $M$ by open subsets $U$ which are finitely generated by a minimal set of generators as in proposition 3.1. Then, one considers the bisubmersions associated to these generators. This gives us a family of path-holonomy bisubmersions covering $M$ for which we can define a holonomy groupoid.

**Remark.** We will revisit the definition of the holonomy groupoid later on, and we will be a bit more precise regarding some technical details. Thus, it is important to note that above definition *does not tell the whole story.*

We now define the suitable equivalence relation mentioned in example 4.5. For this, we need the following definition.

**Definition 4.6.** [AS] Let $(M, \mathcal{F})$ be a foliated manifold. Suppose $\mathcal{U} = \{(U_i, t_i, s_i)\}_{i \in I}$ is a family of bisubmersions of $(M, \mathcal{F})$. We say that a bisubmersion $(V, t_V, s_V)$ is adapted to $\mathcal{U}$ if for each $v \in V$, we can find a bisubmersion $(U_i, t_i, s_i) \in \mathcal{U}$ and a local morphism of bisubmersions $f : V' \to U_i$ (where $v \in V' \subset V$ open).

Using definition 4.6, we get the following definition.

**Definition 4.7.** In the context of definition 4.5, we endow the disjoint union $\bigsqcup_i U_i$ with the equivalence relation given by $U_i \ni u \sim v \in U_j$ if and only if there is a local morphism of bisubmersions $f : U_i' \to U_j$ such that $f(u) = v$.

**Remark.** It is important to note that $U_i$ and $U_j$ in definition 4.7 need not be different.

**Remark.** Notice that it follows from proposition 5.4(3) that this relation is symmetric.

One might wonder why we only required a local morphism of bisubmersions. For this, recall that we used bisubmersions to record local automorphisms in $\mathrm{Aut}(M, \mathcal{F})$. Using this reasoning, we want to identify points of bisubmersions if and only if they record the same local diffeomorphisms. In terms of bisections, this is equivalent to stating that both points carry the same bisection. Since bisections are local objects, it is more appropriate to consider local morphisms.

Notice that this naturally endows $H$ with the quotient topology, making it a topological space. We now show that it is a groupoid.

**Definition 4.8.** Let $(M, \mathcal{F})$ be a foliated manifold with associated holonomy groupoid $H$. Fix an element $h \in H$. Let $u \in (U, t_U, s_U)$ be an element such that $h = [u]$. Then we define

$$s(h) = s_U(u), t(h) = t_U(u).$$

Now let us check whether definition 4.8 makes sense.

**Proposition 4.7.** Let $(M, \mathcal{F})$ be a foliated manifold with holonomy groupoid $H$. Then the structure maps $s, t : H \to M$ are well-defined.

*Proof.* Let $h \in H$, and suppose there are $u \in (U, t_U, s_U)$ and $v \in (V, t_V, s_V)$ such that $h = [u] = [v]$. Let us prove that $s(h) = s_U(u) = s_V(v)$. For this, recall that by definition there exists a local morphism of bisubmersions $f : U' \to V$ such that $f(u) = v$. A morphism of bisubmersions satisfies by definition $s_U(u) = s_V(f(u)) = s_V(v)$. Hence, the map $s$ is well-defined for the equivalence relation given above. The proof for the target map is completely analogous. □

For $H$ to be a groupoid, we still need to define the partially defined multiplication. For this, we will need the following lemma.

**Lemma 4.2.** Let $(U, t_U, s)$ and $(V, t_V, s_V)$ be two bisubmersions. Let $(u, v) \in U \times_{s_U, t_V} V$, which implies $s_U(u) = t_V(v)$. Suppose $\phi$ is carried by $(U, t_U, s_U)$ at $u$ and $\psi$ is carried by $(V, t_V, s_V)$ at $v$. Then, the composition $\phi \circ \psi$ is carried by the composition $U \circ V = (U \times_{s_U, t_V} V, t_V, s_U)$ at $(u, v)$.

*Proof.* By definition, we can find bisections $N_\phi$ and $N_\psi$ at $u$ respectively $v$ for which the germs of the induced diffeomorphisms coincide with $\phi$ respectively $\psi$. Consider the bisection $N := N_\phi \times_{s_U,t_V} N_\psi$. Evidently, $(u, v) \in N$. The diffeomorphism it carries is now given by $t_U|_N \circ s_V^{-1}|_N$. Since $s_U|_N = t_V|_N$, we have that $s_U^{-1}|_N = t_V^{-1}|_N = id_N$. Hence, we find

$$t_U|_N \circ s_V^{-1}|_N = t_U|_N \circ s_U^{-1}|_N \circ t_V|_N \circ s_V^{-1}|_N.$$

By our assumptions, the latter coincides with $\phi \circ \psi$ in a small neighbourhood of $(u, v)$. This proves the claim. □

**Remark.** Here, we abused the notation a bit and considered $s_V$ as the map $(u, v) \mapsto s_V(v)$ and $t_U$ as the map $(u, v) \mapsto t_U(u)$.

Using this, we can define the multiplication as follows.

**Definition 4.9.** On $H$, we define the following multiplication. Let $h_1, h_2 \in H$ with $s(h_1) = t(h_2)$, and suppose that $h_1 = [u_1]$ and $h_2 = [u_2]$, where $u_1 \in (U_1, t_1, s_1)$ and $u_2 \in (U_2, t_2, s_2)$. We then define

$$h_1h_2 = [(u_1, u_2)],$$

where $[(u_1, u_2)]$ is the image of $(u_1, u_2) \in U_1 \circ U_2$ in $H$.

**Proposition 4.8.** This multiplication satisfies the required conditions: it is well-defined and associative.

*Proof.* Let us first check whether it is well-defined. There are several things we need to check. First, notice that $s_1(u_1) = t_2(u_2)$, which follows from lemma 4.7. Hence, $(u_1, u_2) \in U_1 \circ U_2$. Second of all, since $\mathcal{U}$ contains the finite compositions of elements in $\mathcal{U}$, it also makes sense to talk about $[(u_1, u_2)]$. We now need to check if the definition is independent of choice of representatives. Hence, suppose we have another pair $u_1' \in (U_1', t_1', s_1'), u_2' \in (U_2', t_2', s_2)$ satisfying $h_1 = [u_1']$ and $h_2 = [u_2']$. Recall that the existence of a local morphism $f$ (with $f(u) = v$) is equivalent with the two bisubmersions carrying the same diffeomorphism at $u$ and $v$ respectively. Since $[u_1] = [u_1']$ and $[v_1] = [v_1']$, we find that there exists a diffeomorphism $\phi_1$ carried by $U_1$ at $u_1$ and $U_1'$ at $u_1'$, and a diffeomorphism $\phi_2$ satisfying the same statement for $U_2$ and $U_2'$. By lemma 4.2, the compositions carry $\phi_1 \circ \phi_2$ at $(u_1, u_2)$ and $(u_1', u_2')$ respectively. Thus, they carry the same diffeomorphism, implying that $[(u_1, u_2)] = [(u_1', u_2')]$ from which it follows that the multiplication is well-defined. The fact that the multiplication is associative follows from the associativity of the composition of diffeomorphisms. □

Finally, we define the identity elements and show their existence.

**Definition 4.10.** An element $h \in H$ is called an identity element if $h$ carries the identity on $M$. In this case, $s(h) = x$, we denote $h = 1_x$

Just like for compositions, we will need the following lemma for inverses.

**Lemma 4.3.** Suppose $(U, t, s)$ carries a diffeomorphism $\phi$ at $u$. Then the inverse bisubmersion $(U, s, t)$ carries $\phi^{-1}$ at $u$.

**Proposition 4.9.** Given an element $x \in M$, then $1_x \in H$. Furthermore, this element satisfies $h1_x = h = 1_y h$ whenever $s(h) = y, t(h) = x$.

*Proof.* Let $(U, t, s) \in \mathcal{U}$ be a path-holonomy bisubmersion such that $x \in s(U)$. By definition, $(U, s, t)$ (which is the inverse of $(U, t, s)$) also lies in $\mathcal{U}$. Consider an element $u \in (U, t, s)$ such that $s(u) = x$. Denote by $u^{-1} \in (U, s, t)$ the corresponding element in the inverse. We denote by $h^{-1}$ its equivalence class $[u^{-1}]$ in $H$. Then combining lemma 4.2 and lemma 4.3 we find that $h^{-1}h$ carries the identity, and hence $1_x = h^{-1}h$. The multiplication identities are an immediate consequence of lemma 4.2. □

## 4.3.2 Groupoid of an atlas

In this section, we will revisit the definition of a holonomy groupoid, this time being a bit more precise. We will give a more general way to construct a groupoid from a certain choice of bisubmersions.

**Definition 4.11.** Let $\mathcal{U} = (U_i, t_i, s_i)$ be a family of bisubmersions. We call $\mathcal{U}$ an atlas if

1. $\mathcal{U}$ covers $M$,
2. $\mathcal{U}$ has adapted inverses, i.e for all $(U_i, t_i, s_i) \in \mathcal{U}$, the bisubmersion $(U_i, s_i, t_i)$ is adapted to $\mathcal{U}$, and
3. $\mathcal{U}$ has adapted finite compositions.

A choice of atlas (and hence a choice of bisubmersions) corresponds to a choice of 'recorded local automorphisms'. Thus, it is natural to compare atlases. Motivated by what we have found earlier, the following definition makes sense.

**Definition 4.12.** Let $\mathcal{U} = (U_i, t_i, s_i)$ and $\mathcal{V} = (V_i, t'_i, s'_i)$ be two atlases. We say that $\mathcal{U}$ is adapted to $\mathcal{V}$ if every element of $\mathcal{U}$ is adapted to $\mathcal{V}$ (in the sense of definition 4.6). We say that $\mathcal{U}$ and $\mathcal{V}$ are equivalent if they are adapted to each other.

Intuitively, an atlas $\mathcal{U}$ is adapted to $\mathcal{V}$ if all the local diffeomorphisms recorded by $\mathcal{U}$ are also recorded by $\mathcal{V}$.

We now define the groupoid of an atlas, which is a generalisation of the construction we made in the previous section.

**Definition 4.13** (The groupoid of an atlas)**.** Let $\mathcal{U} = (U_i, t_i, s_i)$ be an atlas. We endow the set $\bigsqcup_{U \in \mathcal{U}} U$ with the equivalence relation seen in definition 4.7. We denote by $G_{\mathcal{U}}$ the quotient of this equivalence relation. We furthermore denote by $Q = (q_i)_i : \bigsqcup_i U_i \to G_{\mathcal{U}}$ the quotient map. We endow $G_{\mathcal{U}}$ with the structure maps satisfying

$$\begin{array}{ccc} G & \xrightarrow{s} & M \\ & \searrow^{q_i} & \uparrow{\scriptstyle s_i} \\ & & U_i \end{array} \qquad \begin{array}{ccc} M & \xleftarrow{t} & G \\ {\scriptstyle t_i}\uparrow & \swarrow_{q_i} & \\ U_i & & \end{array}$$

We further endow $G_{\mathcal{U}}$ with a partially defined multiplication given by

$$q_i(u)q_j(v) = q_{U_i \circ U_j}(u, v).$$

For the multiplication to make sense, we used following result.

**Proposition 4.10.** Consider the setting as in above definition. For every bisubmersion $(U, t_U, s_U)$ adapted to $\mathcal{U}$, there exists a map $q_U : U' \to G_{\mathcal{U}}$ such that, for every local morphism $f : U' \to U_i$ and every $u \in U'$, we have $q_U(u) = q_i(f(u))$.

**Remark.** One can show that the groupoid of an atlas is indeed a groupoid. For this, one can use similar strategies as we used in previous section, but they must be more careful since we now don't have compositions but adapted compositions.

Let us consider some examples.

**Example 4.2.** Given a covering by $s-$connected path-holonomy bisubmersions, one can choose $\mathcal{U}$ as the smallest atlas containing said cover. The groupoid associated to this atlas is called a path-holonomy atlas. We will later use this path-holonomy atlas to (re)define the holonomy groupoid.

**Example 4.3.** Consider the leaf-preserving atlas, which consists of all bisubmersions $(U, t, s)$ such that for all $u \in U$, $s(u)$ and $t(u)$ lie on the same leaf. This is an atlas: compositions and inverses of leaf-preserving bisubmersions are evidently again leaf-preserving. Notice that in particular, the path-holonomy bisubmersions are leaf-preserving. Hence, the leaf-preserving atlas covers $M$ (and is hence an atlas). Also, notice that the fact that path-holonomy bisubmersions are leaf-preserving implies that path-holonomy atlases are adapted to the leaf-preserving atlas.

**Example 4.4.** Suppose that $(M, \mathcal{F})$ is a foliated manifold. Suppose $\mathcal{F}$ comes from a Lie groupoid $G \rightrightarrows M$. We have already seen in proposition 4.1 that $(G, t, s)$ is a bisubmersion. But it actually an atlas. The fact that $(G, t, s)$ covers $M$ is immediate: the source map is always a surjective submersion onto the base manifold. Furthermore, $G$ evidently has an adapted inverse and adapted compositions. One can show that if $G$ is $s-$connected, the path holonomy atlas is actually equivalent to $(G, t, s)$. Hence, the path-holonomy atlas is a quotient of this atlas, i.e it is a quotient of $G$.

There is a particularly nice result relating the relation between atlases and the relation between their respective groupoids.

**Proposition 4.11.** Let $\mathcal{U}, \mathcal{V}$ be two atlases. Suppose $\mathcal{U}$ is adapted to $\mathcal{V}$, then there is a natural injective groupoid morphism $\alpha : G_{\mathcal{U}} \to G_{\mathcal{V}}$. Furthermore, $\alpha$ is bijective if and only if the atlases are equivalent.

*Proof.* We sketch the proof. Consider the map $\tilde{\alpha} : \coprod U_i \to G_{\mathcal{V}}$, defined as follows. For any $u \in \mathcal{U}$, let $f$ be a morphism of bisubmersions $f : U' \to V$, where $U'$ is an open neighbourhood of $u$ and $V$ is a bisubmersion in $\mathcal{V}$. Then $\tilde{\alpha}(u) = [f(u)]$. One can show that $\tilde{\alpha}$ induces a well-defined map $\alpha$ on the quotient, and that $\alpha$ is injective. The second part is easy: $\alpha$ is surjective if and only if each $v \in V \in \mathcal{V}$ has an equivalent $u \in U \in \mathcal{U}$, which means $\mathcal{V}$ must be adapted to $\mathcal{U}$. □

We now claim that a path holonomy bisubmersion is adapted to any atlas $\mathcal{V}$.

**Proposition 4.12.** Let $(W, t, s)$ be a path-holonomy bisubmersion. Let $\mathcal{V}$ be *any* atlas of bisubmersions for $\mathcal{F}$. Then $W$ is adapted to $\mathcal{V}$.

*Proof.* Recall that $W$ is an open neighbourhood of $(0, x)$ in $\mathbb{R}^n \times M$. Let $(y, x)$ be any point in $W$ with respect to this decomposition. We need to find a local morphism of bisubmersions $f : W' \to V$ with $W'$ a neighbourhood of $(y, x)$ and $V$ a bisubmersion in the atlas $\mathcal{V}$. Let $X_1, ..., X_n$ be the local generators associated to $W$. Furthermore, consider the vector field $\tilde{X} = \sum_i y_i X_i$. If $y = 0$, the point $(y, x)$ carries the identity diffeomorphism. Since $\mathcal{V}$ is an atlas, consider a bisubmersion $V \in \mathcal{V}$ such that $x \in s(V)$. Since $\mathcal{V}$ has adapted compositions and adapted inverses, it is easy to see that there is a bisubmersion adapted to $\mathcal{V}$ that carries the identity diffeomorphism "at $x$". By theorem 4.1 and proposition 4.6, there is a local morphism $f : W' \to V$. If $y \neq 0$, $\tilde{X} \neq 0$. One can thus find a new set of generators $Y_1, ..., Y_n$ where $Y_1 = \tilde{X}$. These give a new bisubmersion $(W", t, s)$, and it is easy to see that there is a morphism of bisubmersions $f : W \to W$ with $f((y, x)) = ((1, 0, ..., 0), x)$. Since the property of being equivalent is transitive, we may assume $W = W"$ and $(y, x) = ((1, 0, ..., 0), x)$. Consider now the path $\alpha : I \to M$ defined by $\alpha(t) = \exp(tY_1)$. Since $Y_1 \in \mathcal{F}$, this map is an automorphism of the foliation, so along each point of the path applying $\alpha(t)$ to $Y_1, ..., Y_n$ we get a set of generators $Y_1^t, ..., Y_n^t$ of $\mathcal{F}$ in a neighbourhood of $\alpha(t)$. Notice that $Y_1^t = Y_1$. Since this is a set of generators, we can find for each $t$ a bisubmersion $(W_t, s, t)$, with $W_t$ a neighbourhood of $(0, \alpha(t))$. Just like we did above, we can find a morphism of bisubmersions $f_t : W_t' \to V_t$, where $W_t'$ is an open neighbourhood in $W_t'$ of $(0, \alpha(t))$. By choosing $n \in \mathbb{N}$ large enough, we can find a finite ($\alpha(I)$ is compact) covering $W_{\alpha(\frac{i}{n})}$ with the property that $(\frac{1}{n}y, \gamma(\frac{i}{n})) \in W_{\alpha(\frac{i}{n})}$. Notice that in this case, we can compose $(\frac{1}{n}y, \gamma(\frac{i}{n})) \circ (\frac{1}{n}y, \gamma(\frac{i-1}{n}))$. Since this composition carries the diffeomorphism $\exp(Y_1)$, and since the same holds for $(y, x)$, they are equivalent. However, each factor in the composition had an associated bisubmersion $V_{\frac{i}{n}}$ in $\mathcal{V}$. Thus, their composition also has an associated bisubmersion in $\mathcal{V}$, since the latter has adapted compositions. This shows the desired result. □

Combining both results, we obtain the following results.

1. A path holonomy atlas is adapted to any other atlas. Thus, given the groupoid of a path-holonomy atlas $G_{\mathcal{U}}$ and a groupoid of any other atlas $G_{\mathcal{V}}$, one can injectively map $G_{\mathcal{U}} \to G_{\mathcal{V}}$.

2. In fact, given two path-holonomy bisubmersions $\mathcal{V}_1$ and $\mathcal{V}_2$, it follows that $G_{\mathcal{V}_1}$ and $G_{\mathcal{V}_2}$ are isomorphic as groupoids. Therefore, it makes sense to talk about *the* groupoid associated to *the* path-holonomy atlas.

Thus, the following definition makes sense.

**Definition 4.14.** Let $(M, \mathcal{F})$ be a foliated manifold. Let $\mathcal{V}$ be a path-holonomy atlas associated to $\mathcal{F}$. Then the groupoid $H(\mathcal{F})$ associated to the path-holonomy atlas is called the holonomy groupoid.

**Remark.** One could wonder if this notion of holonomy is a generalisation of the one given in the case of regular foliations. Since a regular foliation is also singular, we are able to compare the two. One can show that they do indeed coincide. Let us present the sketch of a proof that shows this. First, recall that $\mathrm{Hol}(M, \mathcal{F})$ (the holonomy groupoid in the regular case) was the minimal (in the sense of Moerdijk-Mrcun [MM2]) Lie groupoid integrating $\mathcal{F}$. Here, One can show that $H$ is the minimal topological groupoid integrating the foliation. Thus, it remains to show that $H$ is a Lie groupoid, in which case minimality of both groupoids forces them to be equal. In section 4.4, we will discuss a result which shows that $H$ is indeed a Lie groupoid in the case of regular foliations.

We now consider some examples of holonomy groupoids.

**Example 4.5.** Consider the foliation of $\mathcal{F}$ on $\mathbb{R}^2$ coming from the action of $S^1$. As we have already mentioned, this foliation is generated by the rotational vector field $x\partial_y - y\partial_x$. Consider now the bisubmersion $S^1 \times \mathbb{R}^2$, whose source map is the second projection and the target map $t(\theta, x) = r_\theta x$ is the rotation by an angle $\theta$. This is nothing else than the action groupoid. As we have already seen in example 4.4, this is indeed a bisubmersion. In fact, since the source fibers are diffeomorphic to $S^1$, this groupoid is $s-$connected. Whence, the holonomy groupoid is diffeomorphic to a quotient of the action groupoid $S^1 \times \mathbb{R}^2$. What we still have to do is study the quotient. First of all, notice that outside of the origin, the foliation is regular. The holonomy groupoid for this regular part of the foliation is isomorphic to $S^1 \times (\mathbb{R}^2 \setminus \{0\})$. Hence,

$$\mathrm{Hol}(\mathcal{F})|_{\mathbb{R}^2 \setminus \{0\}} \cong S^1 \times (\mathbb{R}^2 \setminus \{0\}).$$

Let us now look at $q_0$, the projection at the origin (Here, the projection is the natural projection $S^1 \times \mathbb{R}^2 \to H$). We claim that the map is indeed injective. For this, we need to show that for any two different points $(g, 0)$ and $(g', 0)$, there are bisections through these points that carry different diffeomorphisms. A smooth local section of the action groupoid is a smooth map $x \mapsto (\phi(x), x)$, where $\phi$ is a smooth map from an open neighbourhood of 0 in $\mathbb{R}^2$ to $S^1$. These maps determine fully the local section, and we denote the smooth map associated to a bisection through $g$ respectively $g'$ by $\phi_g$ and $\phi'_g$. Notice that $\phi_g(0) = g$ and $\phi_{g'}(0) = g'$. The associated bisections are given by $x \mapsto \phi_g(x) \cdot x$ and $x \mapsto \phi_{g'}(x) \cdot x$ respectively. These maps are different: their derivatives at 0 are the rotations by the angles of $g$ and $g'$ respectively, and since $g \neq g'$, their derivatives do not coincide. Thus, they carry different bisections which implies that $(g, 0)$ and $(g', 0)$ are not equivalent in the quotient. Thus, the holonomy groupoid coincides with the action groupoid.

**Example 4.6.** Consider the foliation of $\mathbb{R}$ generated by $x\partial_x$. Consider the path holonomy bisubmersion generated by $x\partial_x$, i.e $(U, t, s) \in \mathbb{R} \times \mathbb{R}$ (we only have one generator!) given by $s(z, y) = z$ and $t(z, y) = e^y z$, which is the time-$y$ flow of $x\partial_x$, starting at $z$. This is an atlas, and since it is a path holonomy bisubmersion the holonomy groupoid is an atlas of it. Again, we claim that it is actually isomorphic to the action groupoid. The idea is the same: outside of the origin we have a regular part, and the argument above can be repeated for the singular part in the origin. In figure 4.2, we show the difference between the bisections $V_1$ and $V_2$.

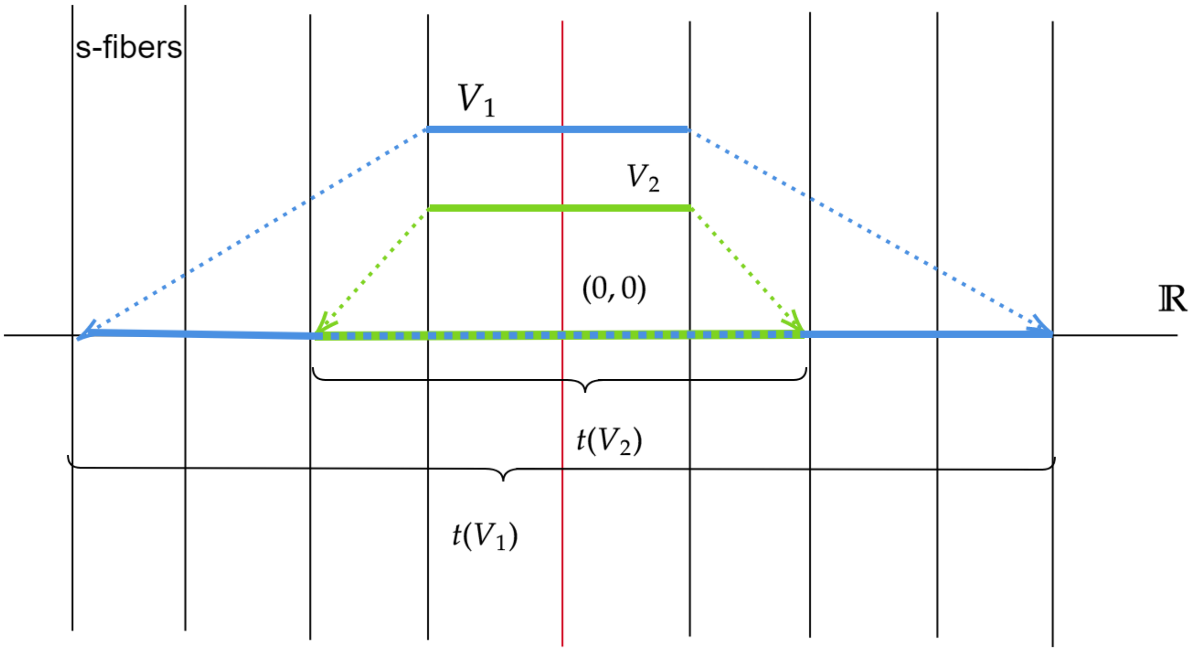



Figure 4.2: A graphical representation of the different bisections

However, it is not always the case that the holonomy groupoid and the action groupoid coincide.

**Example 4.7.** Consider the action of $GL_2(\mathbb{R})$ on $\mathbb{R}^2$. Outside of the origin, it is generated by $\partial_x$ and $\partial_y$. At the origin, we have the generators $x\partial_x, y\partial_y, x\partial_y$ and $y\partial_x$. Outside of the origin, the foliation is regular and in fact is the one-leaf foliation. This regular foliation has as associated holonomy groupoid the pair groupoid, and hence outside of the origin we obtain the pair groupoid

$$\mathrm{Hol}(\mathcal{F})|_{\mathbb{R}^2\setminus\{0\}} \cong \left(\mathbb{R}^2 \setminus 0\right) \times \left(\mathbb{R}^2 \setminus \{0\}\right).$$

At the origin, one can show that

$$\mathrm{Hol}(\mathcal{F})|_0 \cong GL_2(\mathbb{R}) \times \{0\}.$$

Notice that in general the holonomy groupoid of a singular foliation need not be smooth, as shown by example 4.7, where the dimension of the source fiber is either 2 or 4 depending on the chosen base point. Thus, the holonomy groupoid is a topological groupoid but need not be a Lie groupoid. In the following section, we briefly touch upon the smoothness of the holonomy groupoid.

## 4.4 Smoothness of the holonomy groupoid

This section gives a brief overview of some results regarding the smoothness of the holonomy groupoid. We have already briefly touched upon the longitudinal smoothness of the holonomy groupoid. Here, we focus on the smoothness of $H$ as a groupoid. We need the following definition.

**Definition 4.15.** Let $M$ be a smooth manifold. A singular foliation $\mathcal{F}$ is called almost regular if it is a projective module. By Serre-Swan's theorem [Sw], this is equivalent with $\mathcal{F}$ being isomorphic to the space of compactly supported sections of a vector bundle.

Recall that a regular foliation could always be described by a Lie algebroid with injective anchor map. The following proposition motivates the terminology of almost regular foliations.

**Proposition 4.13.** Let $\mathcal{F}$ be an almost regular foliation. Then the associated vector bundle $A \to TM$ is a Lie algebroid with as anchor the evaluation map, which is injective on an open dense subset of $M$.

*Proof.* (Sketch) Since $\mathcal{F}$ is involutive, which means it is closed under the Lie bracket, and since we have the isomorphism $\mathcal{F} \cong C_c^\infty(M, A)$, we can pull back the Lie bracket on $\mathcal{F}$ to a Lie bracket on the sections of $A$. Furthermore, the injection $\mathcal{F} \hookrightarrow \mathfrak{X}(M)$ induces an injective map $C_c^\infty(M, A) \hookrightarrow \mathfrak{X}(M)$. It follows that the anchor map $A \to TM$ as defined in the proposition is injective on a dense open subset of $M$. □

It was shown by Debord (in [CD]) that for these type of foliations, one can find a Lie groupoid that integrates the foliation. In fact, there is a canonical isomorphism $\mathrm{Hol}(\mathcal{F}) \cong G$. In other words, projectiveness of $\mathcal{F}$ is a sufficient condition for $\mathrm{Hol}(\mathcal{F})$. Following result states that it is necessary.

**Proposition 4.14.** [AZ2] Let $(M, \mathcal{F})$ be a singular foliation. Then $\mathrm{Hol}(\mathcal{F})$ is a Lie groupoid if and only if $\mathcal{F}$ is a projective foliation.

This tells us that the presence of singularities gets reflected in the holonomy groupoid: it often possesses an ugly topology, and is only smooth in the nicest of singular foliations. However, it is still possible to get a geometric interpretation of the groupoid, in the form of *holonomy transformations*.

# Chapter 5

# Holonomy transformations

In this section, we give a way to geometrically interpret the holonomy groupoid. Recall that in the regular case, we associated to leafwise paths the germ of a diffeomorphism acting on a transverse section $T$. The idea of holonomy transformations is to extend this idea in a suitable way to the case of singular foliations. Concretely, we will define a map $\Phi$ that maps elements of the holonomy groupoid to (equivalence classes of) germs of diffeomorphisms of transverse sections. In the final section, we will show that $\Phi$ is injective. The main reference of this chapter is the paper by Androulidakis and Zambon ([AZ]).

## 5.1 Preliminaries

Before talking about holonomy transformations, we need some preliminary results. The first result we need is an already mentioned splitting result, and can be found in ([AZ], prop. 1.4).

**Theorem 5.1.** [AZ] Let $(M, \mathcal{F})$ be a foliated manifold, and fix an element $x \in M$. Let $\tilde{S}$ be an embedded submanifold of $M$ such that $T_x\tilde{S} \oplus F_x = T_xM$. We call such an embedded submanifold a <u>slice</u> at $x$. Then there exists an open neighbourhood $W$ of $x$ in $M$ and a diffeomorphism of foliated manifolds

$$(W, \mathcal{F}|_W) \cong (I^k, TI^k) \times (S, \mathcal{F}_S).$$

Here, $k = \dim(F_x)$, $I = (-1, 1)$, $\tilde{S} \cap W =: S$ and $\mathcal{F}_S = i^{-1}(\mathcal{F})$ (where $i : S \hookrightarrow W$), i.e $\mathcal{F}_S$ is the restriction of $\mathcal{F}$ to $S$.

This result gives us a nice way to look at transversals in the case of singular foliations. It tells us that we can endow these transversals with singular foliations, vanishing at the origin. Notice that in the case of regular foliations, we just recover the definition of regular foliations, in which case the foliation on the transversal section is the one whose leaves are points.

**Remark.** Notice that if $F_x$ is zero dimensional for some $x$, then the slices are open neighbourhoods around $x$.

**Example 5.1.** Consider the foliation on $\mathbb{R}^2$ generated by the vector fields $\partial_x, y\partial_y$. The leaf given by $y = 0$ is a singular leaf, and in figure 5.1, we show how the splitting theorem locally decomposes the foliation.

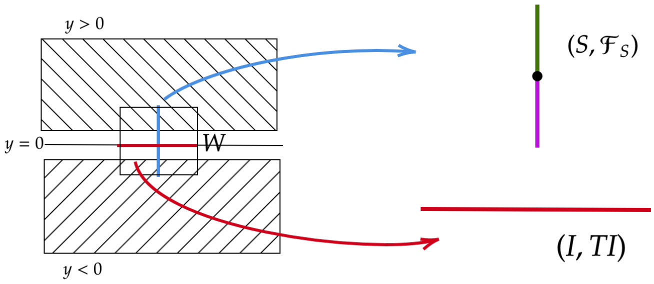



Figure 5.1: The splitting theorem in action

Recall that elements of the holonomy groupoid in the regular case were in particular germs of diffeomorphisms of transversals. Indeed, for $S$ some transversal, we had the map

$$\mathrm{Hol}^S : \pi_1(L, p) \to \mathrm{GermDiff}(S).$$

With this in mind, we want to find a suitable extension to the case of singular foliations. Thus, we want to find a way to associate to elements of the holonomy groupoid a germ of a diffeomorphism of transversals. Since these transversals carry more information than the regular counterparts, the following notations (introduced in [AZ]) are natural.

**Notation 5.1.** Let $(M, \mathcal{F})$ be a foliated manifold, $x, y \in M$ be points on the same leaf. Let $S_x, S_y$ be slices as in the splitting theorem based at $x$ and $y$ respectively. Recall that $\mathcal{F}(x) = \{X \in \mathcal{F} | X(x) = 0\}$ and $I_x = \{f \in C^\infty(M) | f(x) = 0\}$.

1. $\mathrm{GermAut}_{\mathcal{F}}(S_x, S_y)$ is the space of germs of $x$ of foliation diffeomorphisms from $(S_x, \mathcal{F}_{S_x})$ to $(S_y, \mathcal{F}_{S_y})$.
2. $\exp(I_x\mathcal{F})$ is the space of one-time flows of time-dependent vector fields in $I_x\mathcal{F}$. For a small recap regarding time-dependent vector fields, see appendix 6.2. Analogously, one defines $\exp(I_x\mathcal{F}_{S_x})$, $\exp(\mathcal{F}_{S_x})$ and $\exp(\mathcal{F}(x))$.

**Remark.** In the case of regular foliations, we have already seen that $\mathcal{F}_{S_x}$ was the foliation on $S_x$ given by points. Thus, $\exp(I_x\mathcal{F}_{S_x})$ and $\exp(\mathcal{F}_{S_x})$ are given by the time-1 flow of the trivial vector field, and hence equal $\{Id_{S_x}\}$.

## 5.2 Construction

In this section, we construct the desired group morphism. We first translate the construction of holonomy in the regular case to a more suitable setting. Instead of taking the 'leaf-wise' approach, we consider a more vector field theoretic point of view. Let $\alpha : I \to M$ be a leafwise path in $(M, \mathcal{F})$ with $\alpha(0) = x$ and $\alpha(1) = y$. At each time $t \in I$, we extend $\alpha'(t)$ to a vector field $Z^t$ lying in $\mathcal{F}$ in such a way that the flow $\Gamma : S_x \times I \to M$ of the time dependent vector field $Z^t$ takes $S_x$ to $S_y$. We will see later that this coincides with the usual notion of holonomy. As we will show later, this gives us a natural extension to the case of singular foliations.

**Remark.** There is no canonical well-defined map $\pi_1(L, x) \to \mathrm{GermAut}_{\mathcal{F}}(S_x, S_y)$. Indeed, consider the singular foliation by the Lie groupoid action $S^1 \times \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ given by rotations. Consider the leafwise path $\alpha$ which is the constant path in the origin. A slice is an open subset of 0 in $\mathbb{R}^2$. An obvious extension is the time-dependent vector field $Y(x, t) = (x, 0)$ for all $x$. The germ associated to this extension is the identity on this open subset. However, another extension as above is the one associated to the flow of $x\partial_y - y\partial_x$. The associated diffeomorphism corresponds to a rotation. They have different derivatives at 0, and hence obviously define different diffeomorphisms.

The reason this fails in the singular case comes from the fact that we have too much choice in the extension. To look at this more carefully, we look at the following result, whose statement and proof can be found in ([AZ], prop. 2.3). It tells us how the ambiguity manifests itself at the level of the obtained diffeomorphism, namely that different choices induce diffeomorphisms differing by an element in $\exp(\mathcal{F}_{S_x})$.

**Lemma 5.1.** [AZ] Let $\alpha$ be a leafwise path, then the image of $\Gamma(., 1)$ in the quotient

$$\mathrm{GermAut}_{\mathcal{F}}(S_x, S_y)/\exp(\mathcal{F}_{S_x})$$

is well-defined.

*Proof.* We denote $x = \alpha(0)$ and $y = \alpha(1)$. Let $Z_t$ and $Z'_t$ be time-dependent vector fields defining extensions $\Gamma$ and $\Gamma'$ respectively. Denote by $\phi_t$ and $\phi'_t$ their flow maps. One can show ([P]) that the following identity holds

$$\phi'_t = \phi_t \circ \left(\text{time t flow of} \left\{(\phi_s)_*^{-1}(Z'_s - Z_s)\right\}_{s\in\mathbb{R}}\right). \tag{5.1}$$

Since both time-dependent vector fields came from extensions of $\alpha$, they satisfy $(\phi_s)_*^{-1}(Z'_s - Z_s)(x) = 0$ for all $s$. Thus, their time $t$ flow lies, by definition, in the set $\exp(\mathcal{F}(x))$. Denote this time-$t$ flow by $N$. Recall that $\mathcal{F}_{S_x}$ was obtained by restricting $\mathcal{F}$ to $S_x$. Hence, we need to find a vector field in $\mathcal{F}$ tangent to $S_x$ (and thus in $\mathcal{F}_{S_x}$) whose time-$t$ flow coincides with $(\phi_s)_*^{-1}(Z'_s - Z_s)$ on $S_x$. This can be done using a slightly different version of lemma 5.4, which we will see later. □

We now define the notion of holonomy transformations, as given in ([AZ], def. 2.4).

**Definition 5.1.** Let $(M, \mathcal{F})$ be a foliated manifold and let $x, y \in M$ be points on the same leaf. Let $S_x$ and $S_y$ be slices at $x$ and $y$ respectively.
Then a holonomy transformation from $x$ to $y$ is an equivalence class in

$$\frac{\mathrm{GermAut}_{\mathcal{F}}(S_x, S_y)}{\exp(I_x\mathcal{F}_{S_x})}.$$

The set of all holonomy transformations is denoted $HT(\mathcal{F})$, or simply by $HT$ if the foliation is clear from the context.

Notice that the denominator is different than the one given in lemma 5.1.

**Remark.** One might wonder why we choose this finer equivalence relation, when the coarser one works too. An important reason is that for elements in $\exp(\mathcal{F}_{S_x})$, the derivative at $x$ is not necessarily the identity. As an example, consider the rotations in the plane as discussed in remark 5.2.

Let us look at what happens when the leaf is regular.

**Proposition 5.1.** [AZ] Let $(M, \mathcal{F})$ be a foliated manifold, and suppose $F$ is a regular leaf. Then $\exp(I_x \mathcal{F}_{S_x})$ and $\exp(\mathcal{F}_{S_x})$ are trivial.

*Proof.* Recall that $F$ is a regular leaf if and only if there is a neighbourhood $U$ around $F$ such that for each leaf $F'$ satisfying $F' \cap U \neq \emptyset$, one has $\dim F = \dim F'$. Let $T_x$ be a slice at $x$, and consider $S_x = T_x|_{T_x \cap U}$. Then $\mathcal{F}_{S_x}$ is a trivial foliation (i.e a foliation by points) since $S_x \cap U$ is transversal to every leaf it intersects. In this case, $I_x \mathcal{F}_{S_x} = 0$, and hence $\exp(I_x \mathcal{F}_{S_x})$ is trivial. The fact that $\exp(\mathcal{F}_{S_x})$ are trivial follows from the splitting theorem. Notice that for regular foliations, the foliation on $S_x$ as in the splitting theorem is the foliation by points. The associated vector field to this foliation is the trivial vector field, hence $\exp(\mathcal{F}_{S_x}) = Id_{S_x}$ is trivial. □

Thus, holonomy transformations are elements of $\mathrm{Aut}_{\mathcal{F}}(S_x, S_y)$. Recall that a path holonomy was also an element of $\mathrm{Aut}_{\mathcal{F}}(S_x, S_y)$. Thus, we have just shown that in the regular case, the holonomy of a path was a holonomy transformation (which tells us we are looking in the right direction).

Recall furthermore that $H_x^y = \{(x, y, [\alpha])\}$. Hence, we get an injective set map

$$H \to HT.$$

The above map has geometrical value: recall that often one thought about the holonomy of paths by looking at their action on slices. This is exactly what is encoded by the above map. The idea is now to extend this idea to the singular case.

## 5.3 Interpreting the Holonomy Groupoid

In this final section, we give a way to interpret the holonomy groupoid in a geometric way. There are two main statements, theorem 5.2 and theorem 5.3. Briefly, the first theorem tells us that we can interpret elements in the holonomy groupoid via their action on slices, and the second theorem tells us that this is faithful in a certain sense.

### 5.3.1 Main statement

In this section, we extend earlier results to get a more geometrical interpretation of the holonomy groupoid. For this, we follow the approach of Androulidakis and Zambon in section 2.3 of their paper [AZ]. The goal is to find a (well-defined) map

$$\Phi_x^y : H_x^y \to \frac{\mathrm{GermAut}_{\mathcal{F}}(S_x, S_y)}{\exp(I_x \mathcal{F}_{S_x})}$$

for some fixed transversals $S_x$ and $S_y$. The statement we wish to prove is the following.

**Theorem 5.2.** Let $(M, \mathcal{F})$ be a foliated manifold, and choose $x, y \in L$. Let $S_x$ and $S_y$ be slices at $x$ and $y$ respectively. Then, the map

$$\Phi_y^x : H_x^y \to \frac{\mathrm{GermAut}_{\mathcal{F}}(S_x, S_y)}{\exp(I_x \mathcal{F}_{S_x})} : h \mapsto [\xi],$$

defined by taking any representative $u \in (U, t, s)$ of $h$ in the path-holonomy atlas, any section $\bar{b} : S_x \to U$ (satisfying $(t \circ \bar{b})(S_x) \subset S_y$) through u, and setting $\xi = t \circ \bar{b}$ is well-defined.

For a graphical representation of this picture, see figure 5.2

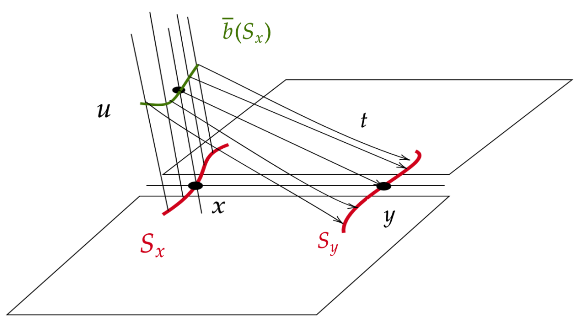



Figure 5.2: A graphical representation of theorem 5.2

As one can see, there are several choices in the construction, and it is far from clear that these leave the map well-defined. Independencies of choices aside, we first check if the map can even exist. We assumed the existence of a bisection carrying a diffeomorphism mapping $S_x$ into $S_y$. The existence is guaranteed by the following result, see ([AZ], lemma A.5) where the full proof can be found.

**Lemma 5.2.** Let $(U, t, s)$ be a bisubmersion and choose $p \in U$. Fix slices $S_x$ and $S_y$ at $x = s(p)$ and $y = t(p)$. Then one can find a bisection through $p$ which carries a diffeomorphism mapping $S_x$ into $S_y$.

*Proof.* We give a sketch of the proof. To find the bisection $\beta$, we first look at a section $\beta : S_x \to s^{-1}(S_x) \cap t^{-1}(S_y)$ whose existence can be argued from the fact that restricting the source map to $s^{-1}(S_x) \cap t^{-1}(S_y)$ gives us (locally) a submersion. For this to make sense, we need to show that the the latter domain is a submanifold, which is done by arguing that $(s, t) : U \to M \times M$ is transverse to $p$ at the submanifold $S_x \times S_y$. One then shows that the image of $\beta$ is transversal to the kernel of (the restriction of) $dt$, hence yielding a diffeomorphism $t \circ \beta : S_x \to S_y$. Then, one extends $\beta$ to an honest section $b$ of $s$ with $T_p(\mathrm{Im}(b)) \cap \ker(d_p t) = 0$. The latter implies that $(d_x b)(T_x L) = \{0\} \times T_x L$. □

Since the path holonomy atlas is an atlas of the foliation, there exists an element $(W, t, s)$ in the path holonomy atlas and an element $w \in W$ with $s(w) = x$ and $t(w) = y$. The following result tells us how these bisubmersions look like.

**Lemma 5.3** ([AZ], lemma A.4)**.** Let $(W, t, s)$ be a bisubmersion in the path holonomy atlas. Then $W$ is isomorphic (as bisubmersions) to a finite composition of path holonomy bisubmersions.

*Proof.* The elements in the path holonomy atlas are precisely those isomorphic to finite compositions of path holonomy bisubmersions and the inverse of path holonomy bisubmersions. Thus, our lemma is equivalent with stating that the inverse of path holonomy bisubmersions are isomorphic (as bisubmersions) to a path holonomy bisubmersion. Recall that the inverse of a bisubmersion $(U, t, s)$ is $(U, s, t)$, which we denote by $U^{-1}$. The isomorphism (interpret it as ”flowing back”) is given by

$$U \to U^{-1} : (y, x) \mapsto (-y, \exp_x(\sum y_i X_i)),$$

where $X_i$ are the local generators associated to $U$. It is immediate that this is indeed an isomorphism of bisubmersions. □

With the existence questions out of the way, let us consider how we will attack this problem. The following lemma will come in handy.

**Lemma 5.4** ([AZ], lemma A.6)**.** Let $(M, \mathcal{F})$ be a foliated manifold, and let $x \in S_x$ be a slice. Suppose $\{Z_t\}$ is a time-dependent vector field on $M$ lying in $I_x\mathcal{F}$ whose time-1 $\psi$ flow maps $S_x$ into $S_x$. Then one can find a time-dependent vector field tangent to $S_x$ in $I_x\mathcal{F}_{S_x}$ whose time$-1$ flow coincides with the restriction of $\psi$ to $S_x$.

*Proof.* (Sketch) Let us briefly discuss the idea. Using the splitting theorem, one can find coordinates such that in a neighbourhood $W$ of $x$ we get $W \cong I^k \times S_x$. Using this, one can define the projection $\pi_{S_x} : W \cong I^k \times S_x \to S_x$. One then restricts the flow $\psi_t$ of $\{Z_t\}$ to $S_x$ for all times $t$, and then projects the flow to $S_x$. This yields a smooth family of diffeomorphisms $\phi_t = \pi_{S_x} \circ \psi_t|_{S_x}$, and defines a time-dependent vector field $Y_t$ on $S_x$ whose time-$t$ flow is $\phi_t$. □

With our goal in mind, we notice that we still miss an important ingredient. We wish to compare bisubmersions, in particular the diffeomorphisms they induce on slices. The above lemma tells us that we should try to associate to the induced diffeomorphisms the flow of a time dependent vector field. One way to obtain time dependent vector fields is via (smooth) families of diffeomorphisms. First, we consider the $N-$fold composition of path holonomy bisubmersions. From the definition of the composition of bisubmersions, one can see a composition as

$$W_2 \circ W_1 = (W_2 \times_{s_2,t_1} W_1),$$

which can be identified with $\{(y_2, y_1, x_1) | (y_1, x_1) \in W_1, (y_2, t_1((y_1, x_1)) \in W_2\}$. The latter can be seen as a subset of $\mathbb{R}^n \times \mathbb{R}^n \times M$. In the case of $N-$fold compositions, we can consider them as an open subset of $\mathbb{R}^n \times ... \times \mathbb{R}^n \times M$, where there are $N$ factors $\mathbb{R}^n$. Using this interpretation, we have the following result. One can find the full proof in ([AZ], pp. 35).

**Lemma 5.5** ([AZ], lemma A.4)**.** Let $W$ be any bisubmersion in the path holonomy atlas, $x \subset M_0$ an open subset in $s(W)$. Write $W \subset \mathbb{R}^n \times ... \times \mathbb{R}^n \times M$, where we made use of lemma 5.3. The graph of any map $b : M_0 \to \mathbb{R}^n \times ... \times \mathbb{R}^n$ satisfying $(d_x b)(T_x L) = 0$ admits a canonical deformation to the zero bisection $\{0\} \times M_0$ by paths of bisections of $W$.

*Proof.* (Sketch) Write $b = (b^N, ..., b^1) : M_0 \to R^n \times ... \times \mathbb{R}^n$. Define the path $b_t$ via:

$$\begin{aligned} &b_t := (0, ..., 0, tb^1) \quad \text{for } t \in [0, 1] \\ &b_t := (0, ..., 0, (t-1)b^2, b^1) \quad \text{for } t \in [1, 2] \\ &... \\ &b_t := ((t - N + 1)b^N, b^{N-1}, ..., b^2, b^1) \quad \text{for } t \in [N - 1, N]. \end{aligned}$$

One can show that the image of $s-$sections $\tilde{b}$ of path holonomy bisubmersions such that $(d_x\tilde{b})(T_x L)$ is the zero bisection are bisections near $x$, and that for these maps the image of $t\tilde{b}$ for $t \in [0, 1]$ are bisections at $x$. Iteratively using these results, we find the desired result. □

Let us now prove theorem 5.2. We use the proof found in ([AZ], pp. 36-37).

*Proof.* [AZ] (*of theorem 5.2*) Let $h \in H_x^y$, and consider an element $(U, t, s)$ in the path holonomy bisubmersion with $u \in U$ such that $h = [u]$. For the remainder of the proof, we identify $U$ with its composition of path holonomy bisubmersions using lemma 5.3. Using lemma 5.2, we can find be a section $\beta$ of $s$ such that $(t \circ \beta)(S_x) \subset S_y$. Extend $\beta$ to a bisection $b$ with $(d_x b)(T_x L) = 0$. Denote by $b_t$ the path in lemma 5.5, deforming $b$ to the zero bisection. Let $(V, t', s')$ be any other bisubmersion in the path holonomy atlas containing a point $v \in V$ with $[v] = h$. Let $\alpha : S_x \to V$ be an $s'-$section for which $(t \circ \alpha)(S_x) \subset S_y$. Since $[v] = [u]$, we can find by definition a morphism $f : V \to U$ mapping $v$ to $u$. Evidently, a morphism maps $s'-$sections to $s-$sections, and hence we obtain an $s-$section $f \circ \alpha$ through $f(v) = u$ which records the same diffeomorphism as $\alpha$. Extend $f \circ \alpha$ to bisection $a$ of $U$ through $u$ with the property that $(d_x a)(T_x L) = \{0\} \times T_x L$. Again, we can associate a path $a_t$ of bisections deforming $a$ to the zero bisection. In this way, we only have to work with $U$, or in other words we may forget $V$.

Observe that $f \circ \alpha = a|_{S_x}$ carries the same diffeomorphism as $\alpha$. Furthermore, since each $b_t$ and each $a_t$ is a bisection, we obtain two smooth families of (local) diffeomorphisms $\phi_t$ and $\psi_t$. Consider now the smooth family of (local) diffeomorphisms $\left\{\phi_t^{-1} \circ \psi_t\right\}_{t\in[0,N]}$.

We claim that this is the flow of a time dependent vector field, lying in $I_x\mathcal{F}$. Write $U = W^N \circ ... \circ W^1$, where each $W$ is a path-holonomy bisubmersion. We associate to the smooth family of diffeomorphisms $\{\phi_t\}$ the time-dependent vector field $Z_t$, coming from

$$Z_t(\phi(t)) = \frac{d}{dt}\phi_t(z).$$

Fix an element $1 \le j \le N$, and consider the path-holonomy bisubmersion corresponding to $W^j$. Fix $t \in [0, N]$ corresponding to $W^j$, i.e $t \in [j-1, j]$. Furthermore, let $Y_1^j, ..., Y_n^j$ be the local generators of $\mathcal{F}$ corresponding to $W^j$. Recall how $b_t$ was defined in lemma 5.5, and write $b^j = (b_1^j, ..., b_n^j) : M_0 \to \mathbb{R}^n$. Then

$$Z_t(\phi(z)) = \sum_{i=1}^{n} b_i^j(z) Y_i^j(\phi_t(z)).$$

Analogously, for $a_t$ we find

$$Z_t'(\psi_t(z)) = \sum_i a_i^j(z) \cdot Y(\psi_t(z)).$$

We rewrite both vector fields as follows:

$$Z_t = \sum_i ((\phi_t^{-1})^* b_i^j) \cdot Y_i^j,$$
$$Z_t' = \sum_i ((\psi_t^{-1})^* a_i^j) \cdot Y_i^j.$$

Then their difference can be written as

$$Z_t' - Z_t = \sum_i \underbrace{\left((\psi_t^{-1})^* a_i^j - (\phi_t^{-1})^* b_i^j\right)}_{d_{i,t}} \cdot X_i^j.$$

A difference of time-dependent vector fields is interesting, since it appears in equation (5.1). Recall that this formula tells you how the flows of $Z$ and $Z'$ are related, in terms of their difference $Z' - Z$.

$$\psi_t = \phi_t \circ \left(\text{time-}t \text{ flow of } \left\{(\phi_s)_*^{-1}(Z_s' - Z_s)\right\}_{s\in[0,N]}\right)$$

The claim holds if we can show that the term in curly brackets lies in $I_x\mathcal{F}$. This holds if $d_{i,t}$ vanishes at $\phi_t(x)$ for every $i, t$. Fix $s$ and let $j \in \{1, ..., N\}$ be such that $ts \in [j-1, j]$. Notice that $b(x) = a(x) = u$. Again, by construction, we also have $\psi_s(x) = \phi_s(x)$. From this, it follows that $d_{i,s}$ vanishes at $\phi_s(x)$. Hence, $\phi^* d_{i,s}$ vanishes at $I_x$, from which the claim follows. By construction, $\phi_N(S_x) = (t \circ b)(S_x) \subset S_y$, and the same holds for $\psi_N$. Thus, we can apply lemma 5.4 to conclude. □

This map hence gives us a way to view elements of the holonomy groupoid via their action on slices. Some questions arise: first of all, do we lose any information? The usefulness of this construction is determined by the kernel of this map: if a lot of different elements in the holonomy groupoid have the same image, we lose a lot of crucial information. Second of all, what are its dependencies?

### 5.3.2 Dependencies of Holonomy Transformations

In this section, we show that the construction depends on the choice of module, but (up to some equivalence) is independent of chosen slices. We first tackle the choice of module. We show the dependency on the module via an example.

**Example 5.2.** [AZ] Suppose we have two different foliations $\mathcal{F}, \mathcal{F}'$ whose induced partition on $M$ coincide. We have already seen such an example: consider $M = \mathbb{R}$ with leafs $\{0\}$, the positive and the negative real numbers. These are the leafs of the foliations generated by $x\partial_x$ and $x^2\partial_x$. Outside of the singular leaf, slices are singletons. For the more interesting case, consider the origin. The slice is an open neighbourhood $S_0$ of the origin. Let $(U, t, s)$ be the path-holonomy bisubmersion associated to $\mathcal{F}$, generated by $x\partial_x$. Let $h$ be any element in $H_0^0$, which corresponds to an element $(\lambda, 0) \subset \mathbb{R} \times M$. Since the choice of bisection through $h$ is immaterial, we choose the constant bisection. We already know the diffeomorphism: it will be $x \mapsto \lambda e^x$. Thus, the image of $(\lambda, 0)$ under $\Phi_0^0$ is the equivalence class of this diffeomorphism in $\mathrm{GermAut}_{\mathcal{F}}(S_0, S_0)/\exp(I_0 X)$. Now, let us consider the case $\mathcal{F}'$ generated by $x^2\partial_x$. The constant bisection through $(\lambda, 0)$ now carries the map $\exp_x(\lambda x^2)$. Let us look at what this diffeomorphism looks like.
This diffeomorphism is the time-1 flow of the integral curve of $\lambda x^2\partial_x$. The integral curve is given by the following differential equation.

$$\begin{cases} \alpha(0) = x \\ \alpha'(t) = \lambda\alpha(t)^2. \end{cases}$$

Separating the equation, we find as a solution

$$\alpha(t) = \frac{1}{-\lambda t - k},$$

where $k$ can be explicitly found using the initial value ($k = -\frac{1}{x}$). We find that the diffeomorphism is then given by $x \mapsto \frac{x}{1-\lambda x}$. Thus, $(\lambda, 0)$ will get mapped to the equivalence class of this diffeomorphism.

We now look at the dependency on slices. We state the result, whose proof can be found in ([AZ], lemma 2.12). We will first need following lemma, whose statement and proof can be found in ([AZ], lemma A.9 (pp. 39)).

**Lemma 5.6** ([AZ])**.** Let $(M, \mathcal{F})$ be a foliated manifold, and fix two points $x, y$ on the same leaf. Consider two sets of transversals $S_x, S_y$ and $T_x, T_y$. Then there is a canonical identification

$$\mathrm{GermAut}_{\mathcal{F}}(S_x, S_y)/\exp(I_x\mathcal{F}_{S_x}) \to \mathrm{GermAut}_{\mathcal{F}}(T_x, T_y)/\exp(I_x\mathcal{F}_{T_x}) : [\eta] \mapsto [\vartheta_y \circ \eta \circ \vartheta_x],$$

where $\vartheta_x$ is the restriction of a map $\theta_x \in \exp(I_x\mathcal{F})$ that maps $T_x$ to $S_x$, and analogously $\vartheta_y$ is the restriction of a map $\theta_y \in \exp(I_x\mathcal{F})$ mapping $S_y$ to $T_y$.

**Proposition 5.2.** [AZ] Assuming the setting of above lemma, the following diagram commutes.

$$\begin{array}{ccc} & H_x^y & \\ \swarrow & & \searrow \\ \frac{\mathrm{GermAut}_{\mathcal{F}}(S_x,S_y)}{\exp(I_x\mathcal{F}_{S_x})} & \longrightarrow & \frac{\mathrm{GermAut}_{\mathcal{F}}(T_x,T_y)}{\exp(I_x\mathcal{F}_{T_x})} \end{array}$$

Here, the map between the sets of holonomy transformations is the identification from lemma 5.6.

Thus the choice of transversals is immaterial.

### 5.3.3 Injectivity

Consider the map $\Phi_x^y : H_x^y \to \frac{\mathrm{GermAut}_{\mathcal{F}}(S_x,S_y)}{\exp(I_x\mathcal{F}_{S_x})}$. This identifies an element in the holonomy group with its action on the corresponding slices. Bringing everything together, we have the following result.

**Lemma 5.7.** [AZ] Let $(M,\mathcal{F})$ be a foliated manifold. For every $x \in M$, let $S_x$ be a slice at $x$. Then the map

$$\Phi : H \to \bigcup_{x\sim y} \frac{\mathrm{GermAut}_{\mathcal{F}}(S_x,S_y)}{\exp(I_x\mathcal{F}_{S_x})},$$

defined via theorem 5.2 is a groupoid morphism. Here $x \sim y$ means $x$ and $y$ lie on the same leaf

The main result of this subsection says that above morphism is actually *injective*. Thus, one can really view theorem 5.2 as a geometric interpretation of the holonomy groupoid. Following lemma simplifies the problem at hand.

**Lemma 5.8.** Let $\Phi : H \to G$ be a morphism of groupoids, covering the identity (see the appendix, section 6.3.1). Then if each map $\Phi_x^x : H_x^x \to G_x^x$ is injective, so is $\Phi$.

*Proof.* Let $h \in H$, and suppose $h : x \mapsto y$. Then $\Phi(h) : x \mapsto y$ is an element of $G_x^y$. Suppose $\Phi(h) = \Phi(h')$ for some $h' \in H$. Consider $\Phi(h \circ (h')^{-1})$. The latter lies in $G_x^x$. Notice that $\Phi_x^x(h \circ (h')^{-1}) = \Phi_x^y(h) \circ \Phi_y^x((h')^{-1}) = \Phi_x^y(h) \circ (\Phi_x^y(h'))^{-1}$. Since their images coincide, the latter is $1_x$, but then by injectivity of $\Phi_x^x$, this can only hold if $h \circ (h')^{-1} = 1_x$. The latter implies that $h = h'$, since inverses are unique. □

This reduces the problem to showing that the maps $\Phi_x^x$ are injective, for each $x \in M$. To achieve this, we will need the following lemma which brings local diffeomorphisms in $\exp(I_x\mathcal{F})$ in the context of bisubmersions in a very nice way.

**Lemma 5.9** ([AZ], lemma 2.13)**.** Let $(M,\mathcal{F})$ be a foliated manifold and fix $x \in M$. Suppose $\psi \in \exp(I_x\mathcal{F})$. Then, one can find a path-holonomy bisubmersion $(U,t,s)$ at $x$ and a bisection $b$ through $(x,0)$ carrying the diffeomorphism $\psi$.

*Proof.* Let $X_1, ..., X_n$ be local generators of $\mathcal{F}$ whose image in $\mathcal{F}_x$ form a basis of $\mathcal{F}_x$. Let $V$ be an open neighbourhood around $x$ in which $X_1, ..., X_n$ form a generating set. Let $(U,t,s)$ be the path holonomy bisubmersion obtained from $X_1, ..., X_n$. Since the $X_i$ span $\mathcal{F}$ in $V$, we can find locally defined functions $f_i^s \in I_x$, defined for $s \in [0,1]$, such that if we define $X = \sum f_i^s X_i$ one has $\psi = \exp(X)$ (actually, $\psi$ will be a (finite) composition of such time-1 flows, but one can just repeat the argument for the more general case).

Using the fact that $t$ is a submersion and that $t^{-1}(\mathcal{F}) = C_c^\infty(U, \ker ds) + C_c^\infty(U, \ker dt)$, we can find $Y_1, ..., Y_n$ in $C_c^\infty(U, \ker ds)$ such that $dt(Y_i) = X_i$. Furthermore, we can choose these such that they are linearly independent of each other. Consider the vector field $Y = \sum (f_i^s \circ t) Y_i$, defined for $s \in [0,1]$. Then $X$ and $Y$ are $t-$related, whence

$b(y) := \exp_{(y,0)}(Y)$ (which is a section of $s$ by construction) satisfies $t \circ b = \exp(X) = \psi$. Thus, $b$ is a bisection carrying $\psi$. Notice that for $x$, since all $f_i^s$ lie in $I_x$ and since $t(x,0) = x$, we have that the bisection passes through $(x, 0)$. □

Let us first consider what happens in the singular case, where the foliation vanishes.

**Lemma 5.10** ([AZ], lemma 2.14)**.** Let $(M, \mathcal{F})$ be a foliated manifold and suppose $\mathcal{F}$ vanishes at $x$, i.e $F_x = 0$. Then $\Phi_x^x$ is injective.

*Proof.* Let $h \in H_x^x$, and choose a representative $u \in (U, t, s)$. Suppose $\Phi_x^x(h) = [Id_{S_x}]$, which means that we can find an $s-$section through $u$ carrying $\psi \in \exp(I_x\mathcal{F}_{S_x})$. Notice that, since $F_x$ is zero, $S_x$ must be the same dimension of $M$ (i.e $S_x$ is an open neighbourhood around $x$ in $M$). Hence, $\psi$ is in fact an element of $\exp(I_x\mathcal{F})$. By lemma 5.9, we can find a path-holonomy bisubmersion $(V, t_V, s_V)$ carrying $\psi$ at $(x, 0)$. We can therefore find a morphism of bi-submersions $U \to V$ mapping $u$ to $(x, 0)$. Thus, $h = [u] = [(x, 0)]$, and since $[(x, 0)] = 1_x$, we conclude. □

In ([AZ], section 2.4.1), the authors have shown that a similar statement holds for the regular case.

**Proposition 5.3** ([AZ], prop. 2.17)**.** Let $(M, \mathcal{F})$ be a manifold with a regular foliation. Fix $x \in M$ and a slice $S_x$. Then the map

$$\Phi_x^x : H_x^x \to \text{GermAut}_{\mathcal{F}}(S_x, S_x)$$

is an injective map, and hence, $\Phi_x^x$ is injective.

The reason why lemma 5.10 and lemma 5.3 are of particular interest, is due to the splitting theorem. Recall that this theorem told us that we could decompose a singular foliation in two parts: a regular part and a part where the singular foliation vanishes at a point. The idea is now to reduce the problem to these parts, and use above results. For this, we will need following lemma.

**Lemma 5.11** ([AZ])**.** Let $x \in (M, \mathcal{F})$ and fix an element $h \in H_x^x$. Choose an element $u \in (U, t_U, s_U)$ such that $h = [u]$, and take any $s_U-$section $\bar{b} : S_x \to U$ whose induced diffeomorphism $\tau := t_U \circ \bar{b}$ maps $S_x$ into itself. Let $W$ be a neighbourhood of $x$ and choose a local splitting $W \cong I^k \times S_x$ using the splitting theorem. Then we can find a bisection $b : W \to U$ through $u$ whose induced diffeomorphism is the trivial extension of $\tau$ to every slice:

$$I^k \times S_x \to I^k \times S_x : (s, p) \mapsto (s, \tau(p)).$$

*Proof.* See ([AZ], lemma 2.19). □

Using this lemma, we can state the result and proof as given by Androulidakis-Zambon in ([AZ], thm. 2.20).

**Theorem 5.3.** Let $(M, \mathcal{F})$ be a foliated manifold. Let $x \in M$ and fix a slice $S_x$ through $x$. Then the map

$$\Phi_x^x : H_x^x \to \frac{\text{GermAut}_{\mathcal{F}}(S_x, S_x)}{\exp(I_x\mathcal{F}_{S_x})}$$

is injective. Therefore, $\Phi$ is injective.

*Proof.* As before, we need to show that the only element $h \in H_x^x$ such that $\Phi(h) = [Id_{S_x}]$ is $h = 1_x$. For this, suppose $h$ is any element in $H_x^x$ that gets mapped to the identity. Fix a path-holonomy bisubmersion $(U, t, s)$ such that $u \in U$ satisfies $[u] = h$. Let $\bar{b} : S_x \to U$ be any $s-$section such that $\tau = t \circ \bar{b}$ maps $S_x$ to itself. Then it follows from the assumption that $\tau \in \exp(I_x\mathcal{F}_{S_x})$. Let $\{Y_t\}_{t\in[0,1]}$ be the time-dependent vector field on $S_x$ lying in $I_x\mathcal{F}_{S_x}$ associated to $\tau$. Choose a local splitting $W \cong I^k \times S_x$ of a neighbourhood $W$ of $x$. Let $b : W \to U$ be the bisection through $u$ obtained by lemma 5.11. Then the diffeomorphism $\tilde{\tau} := t \circ b$ trivially extends $\tau$ along every vertical slice. Thus, $\tilde{\tau}$ is the time-one flow of a time-dependent vector field on $W$ obtained via extending $Y_t$ trivially to every vertical slice, from which it follows that $\tilde{\tau} \in \exp(I_x\mathcal{F})$. From lemma 5.9, it follows that any path-holonomy bisubmersion $V$ defined near $x$, the element $(x, 0)$ carries the diffeomorphism $\tilde{\tau}$. This is also the diffeomorphism carried by $b$, and hence we can find a morphism of bisubmersions $U \to V$ mapping $u$ to $(x, 0)$, hence proving the claim. □

# Conclusion

This thesis worked towards answering the three questions posed in the introduction:

1. How is the holonomy groupoid of a regular foliation constructed?

2. Can this construction be generalised to the more general case of singular foliations?

3. Is there a way to geometrically interpret these holonomy groupoids?

In chapter 1, we introduced the notion of holonomy for regular foliations. We saw how this measured how leaves globally twisted around each other. By stating the Reeb stability theorem (theorem 1.1), we saw how this notion could be used to prove some interesting results. In chapter 2 (definition 2.5), we answered the first question: we defined the holonomy groupoid for the regular foliations. In proposition 2.4, we saw that this groupoid had a smooth structure, making it a Lie groupoid.

In chapter 4, we answered the second question. Following the approach by Androulidakis and Skandalis ([AS]), we used the notion of path-holonomy bisubmersions to construct the holonomy groupoid in the singular case. We saw that this groupoid generalises the holonomy groupoid in chapter 2. Furthermore, in section 4.4, we looked at the smoothness question: when is this groupoid a Lie groupoid.

Finally, in chapter 5, we answered the last question. We used the notion of holonomy transformations (as given by Androulidakis and Zambon in [AZ]) to understand how one could geometrically interpret these holonomy groupoids.

# Chapter 6

# Appendix

## 6.1 Appendix A: Frobenius' Lemma

We first need some preliminary definitions.

**Definition 6.1.** A <u>distribution</u> is a subbundle $D \subset TM$.

Thus, a distribution is a smooth choice of linear subspace of constant dimension at each point $p$, varying smoothly over the manifold. Central to Frobenius' lemma is the notion of involutive distributions.

**Definition 6.2.** Let $D \subset TM$ be a distribution on $M$. Denote by $\Gamma(D)$ the smooth sections of $D$. We say that $D$ is <u>involutive</u> if $\Gamma(D)$ is closed under the Lie bracket.

**Remark.** This definition makes sense: as a subbundle of the tangent bundle, the sections of $D$ may be seen as vector fields on $M$ attaining values in $D$. Hence, we can talk about the Lie bracket of sections of $D$.

Given an immersed submanifold $S \subset M$, its tangent bundle can be seen as living in $TM$. In the other direction, suppose $D \subset TM$ is a distribution such that at each point $p \in M$, we can find an immersed submanifold $S_p$ through $p$ satisfying $T_qS = D_q$ for all $q \in M$. Such distributions are called *integrable*, and play a central role in Frobenius' lemma. An immediate question is whether non-integrable distributions exist. The answer is yes, an example is given in example 1.5. Thus, the next natural question is if we can easily determine whether a distribution is integrable. This is the content of Frobenius' lemma:

**Lemma 6.1.** Let $M$ be a smooth manifold, and $D \subset TM$ a distribution on $M$. Then $D$ is integrable if and only if $D$ is involutive.

To conclude this section, we consider the following important result.

**Proposition 6.1.** There is a one-to-one correspondence between foliations on $M$ and involutive distributions on $M$.

$$\{\text{Regular foliations on M}\} \Longleftrightarrow \{\text{Involutive distributions on M}\}.$$

We briefly sketch how this correspondence looks like. Given an involutive distribution, one obtains a foliation by considering through each point $p \in M$ the integral manifolds. Considering the largest (with respect to inlcusion) integral manifold of $D$, one ends up with a foliation. In the other direction, a foliation induces a regular foliation by considering its tangent bundle.

## 6.2 Appendix B: Time dependent vector fields

**Definition 6.3.** A time dependent vector field on a manifold $M$ is a smooth map

$$Y : \mathcal{D} \subset \mathbb{R} \times M \to TM : (t, x) \mapsto Y(t, x) = Y_t(x) \in T_x M.$$

**Remark.** Fixing $t \in \pi_1(\mathcal{D})$ (where $\pi_1$ is the first projection), the vector field $Y(t, -) = Y_t$ is a vector field in the usual sense, defined on the open set $\pi_2(\mathcal{D})$.

**Example 6.1.** Consider the fluid velocity vector field of some unsteady flow of water around a pole.

These vector fields can be seen as honest vector fields on $\mathbb{R} \times M$. Indeed, we define a vector field $\tilde{Y}$ on $\mathbb{R} \times M$ from the time dependent vector field $Y$ by defining $\tilde{Y}(t, x) = (1, Y(t, x))$. In other words, we have $\tilde{Y} = \partial_t + Y$.
Time-dependent vector fields are closely related to isotopies.

**Definition 6.4.** An isotopy of $M$ is a smooth 1−parameter family $\{\phi_t : M \to M\}_{t\in[0,1]}$ of diffeomorphisms such that $\phi_0(x) = x$ for all $x$, and $\Phi : M \times I \to M : (p, t) \to \phi_t(p)$ is smooth.

Any isotopy $\phi_t$ defines a (unique) time-dependent vector field $X_t$ by the equation

$$\frac{d}{dt}\phi_t = \phi_t^* X_t.$$

The vector $X_t(p)$ is the velocity vector of the curve $s \mapsto \phi_s(q)$ at time $t$ where $p = \phi_s(q)$, see figure 6.1

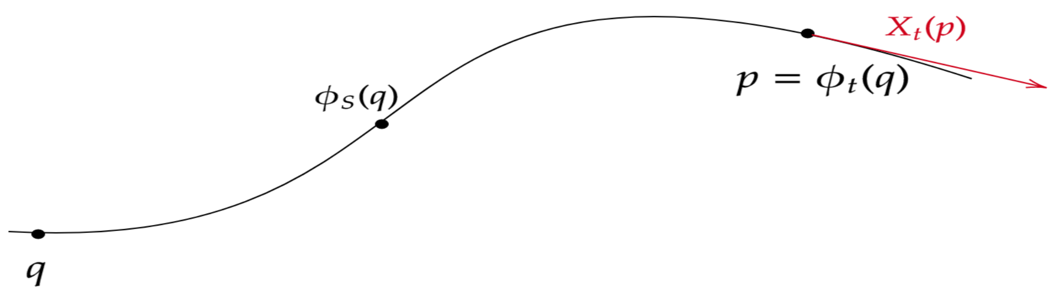



Figure 6.1: The time-dependent vector field of an isotopy

In the other direction, if $X$ is complete, one can integrate a time-dependent vector field to an isotopy $\{\phi_t\}$, similar to the case of time-independent vector fields.

## 6.3 Appendix C: Lie Groupoids and Algebroids

In this part of the appendix, we discuss results regarding groupoids and algebroids. We build upon the material defined in chapter 2.

### 6.3.1 Morphisms of Lie Groupoids

**Definition 6.5.** Let $G \rightrightarrows M$ and $H \rightrightarrows N$ be two Lie groupoids. A morphism of Lie groupoids $(F, f)$ is a pair of smooth maps $F : G \to H$ and $f : M \to N$ such that

1. The following diagrams commute:

$$\begin{array}{ccc} G & \xrightarrow{F} & H \\ \downarrow s & & \downarrow s \\ M & \xrightarrow{f} & N \end{array} \qquad \begin{array}{ccc} G & \xrightarrow{F} & H \\ \downarrow t & & \downarrow t \\ M & \xrightarrow{f} & N \end{array}$$

2. $F$ behaves nicely under multiplication: for any composable $(g, g')$, one has

$$F(g \cdot g') = F(g) \cdot F(g').$$

3. $F$ preserves the identity arrows:

$$F(1_p) = 1_{f(p)} \quad \forall p \in M.$$

**Remark.** Two Lie groupoids are said to be *isomorphic* if there exist Lie groupoid homomorphisms $(F, f)$ and $(G, g)$ such that $F = G^{-1}$ and $f = g^{-1}$. Notice that, in particular, the base manifolds are diffeomorphic.

We would like to remark that this is not the only notion of equivalence of Lie groupoids. To hint to the other cases, recall that a Lie groupoid could be seen as a smooth category in which every arrow is invertible. Thus, one could use the different types of equivalences for categories to define other notions of equivalence of Lie groupoids.
Suppose now that $H \rightrightarrows M$ and $G \rightrightarrows M$ are two Lie groupoids with the same associated base manifold $M$. A Lie groupoid homomorphism is then a pair $(F, f)$, where $f : M \to M$ is smooth. We call $F$ a Lie groupoid homomorphism *covering* $f$.

### 6.3.2 Integrability of Lie Algebroids

In this subsection, we will very briefly touch upon the topic of integrability of Lie algebroids. This section should be seen as an exposition of the main results, and is not meant to be explanatory in any way.
Recall the following classical result in differential geometry, called Lie's third theorem.

**Theorem 6.1.** Every finite-dimensional real Lie algebra $\mathfrak{g}$ integrates to a Lie group $G$.

From this, the integrability of Lie algebroids over a point (which are Lie algebras) follows. This is not the only type of Lie algebroids whose integrability was known: as we have seen in section 4.4, Lie algebroids with almost injective anchor map are integrable by Lie groupoids, one of which corresponds to the holonomy groupoid of the induced foliation. The natural question is then if Lie's third theorem also holds in the case of Lie algebroids. In 1967, Pradines claimed that this was true. This claim was false, however, as Almeida and Molino constructed a counterexample in 1985 (see [AM]). We very briefly sketch the counter-example, but the argument used is more modern. To any closed 2−form $\omega \in \Omega^2_{cl}(M)$, one can associate a certain Lie algebroid $A_\omega$ which as a bundle is $TM \times \mathbb{R}$. The Lie algebra structure on its sections is given by

$$[(u, f), (v, g)]_{A_\omega} = ([u, v]_{TM}, u \cdot g - v \cdot f + \omega(u, v)).$$

To this Lie algebroid, they associated a group $N_x(A_\omega)$. They showed that if $N_x(A_\omega)$ was not discrete, the associated Lie algebroid was not integrable.
There were other counter examples known, but an unanswered question was if there was a computational way to determine if a given Lie algebroid is integrable or not. In [CF2], the authors found the exact obstructions to integrability. For completeness, we state this result.

**Theorem 6.2.** A Lie algebroid $A$ on $M$ is integrable if and only if

- $N_x(A) \subset A_x$ is discrete (i.e $r(x) \neq 0$)
- $\liminf_{y \to x} r(y) > 0$,

for all $x \in M$.

Here, $r$ is a function that measures the discreteness of the groups $N_x(A)$. In the example given above, the first condition was violated.

**Example 6.2.** For completeness, we give a concrete example. Choose $M = S^2 \times S^2$, and consider the 2−form $dV \oplus \lambda dV$. Here, $dV$ is the volume form on $S^2$, and $\lambda \in \mathbb{R}$. Then the Lie algebroid is integrable precisely when $\lambda$ is rational.

### 6.3.3 Singular foliations and Lie algebroids

We have seen in example 3.4 that a Lie algebroid induces a singular foliation, namely the image of the sections under the anchor map. In this section, we consider the converse: whether a singular foliation always has an associated Lie algebroid, as was asked by Androulidakis and Zambon. Before giving the example, let us first consider their argument. Let $\mathcal{F}$ be a foliation coming from a Lie algebroid. Thus, $\mathcal{F} = \#(C^\infty(M, A))$. Consider $\ker(\#_x)$, which is the isotropy of the Lie algebroid $A$ at a point $x$. In the paper, they show that there exists a surjective linear map $\ker(\#_x) \to \mathfrak{g}_x$. Thus, the dimension of the isotropy Lie algebra $\mathfrak{g}_x$ is bounded above by the (vector bundle) rank of $A$. In particular, $\dim(\mathfrak{g}_x)$ is bounded above by some natural number for all $x \in M$.

**Example 6.3.** Using above reasoning, it suffices to find a singular foliation whose isotropy lie algebra is not bounded by above. First, consider the module $\mathcal{F}^k$ of vector fields that vanish at the origin to order $k$. This module consists of vector fields of the form $P(x,y)\partial_x + Q(x,y)\partial_y$, where $P, Q$ are homogeneous polynomials of total degree $k$. Hence, they are generated (using $C^\infty(M)-$linear combinations) by the $2k+2$ vector fields $x^i y^j \partial_x$ and $x^i y^j \partial_y$, where $i + j = k$. This singular foliation thus has $\mathcal{F}^k_{(0,0)} = \mathfrak{g}_{(0,0)} = \mathbb{R}^{2k+2}$. For $k \geq 1$, we now define $\tilde{\mathcal{F}}^k$ , generated by $(x-k)^i y^j \partial_x$ and $(x-k)^i y^j \partial_y$. Intuitively, these are shifts of the $\mathcal{F}^k$ to the point $(k, 0)$. We now glue together these foliations by considering the foliation $\mathcal{F}$ generated by $\cup_{k\geq 1} \phi_k \tilde{\mathcal{F}}^k$, where the $\phi_k$ are smooth bump functions with small support containing $(k, 0)$. We end up with a foliation that for each $k$ behaves like $\mathcal{F}^k$ near the point $(k, 0)$. Thus, the infinitesimal isotropy cannot be globally bounded above, which shows that $\mathcal{F}$ cannot come from a Lie algebroid.

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